National Science Olympiad 2019

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Day 1

1 Given that n and r are positive integers.
Suppose that

$$1+2+\cdots+(n-1)=(n+1)+(n+2)+\cdots+(n+r)$$

Prove that n is a composite number.

2 Given 19 red boxes and 200 blue boxes filled with balls. None of which is empty.
Suppose that every red boxes have a maximum of 200 balls and every blue boxes have a maximum of 19 balls.
Suppose that the sum of all balls in the red boxes is less than the sum of all the balls in the blue boxes.
Prove that there exists a subset of the red boxes and a subset of the blue boxes such that their sum is the same.

3 Given that $ABCD$ is a rectangle such that $AD > AB$, where E is on AD such that $BE \perp AC$.
Let M be the intersection of AC and BE. Let the circumcircle of $\triangle ABE$ intersects AC and BC at N and F.
Moreover, let the circumcircle of $\triangle DNE$ intersects CD at G.
Suppose FG intersects AB at P.
Prove that $PM = PN$.

4 Let us define a *triangle equivalence* a group of numbers that can be arranged as shown

$a + b = c$ $d + e + f = g + h$ $i + j + k + l = m + n + o$
and so on...

Where at the j-th row, the left hand side has $j+1$ terms and the right hand side has j terms.

Now, we are given the first N^2 positive integers, where N is a positive integer. Suppose we eliminate any one number that has the same parity with N.

Prove that the remaining $N^2 - 1$ numbers can be formed into a *triangle equivalence*.

For example, if 10 is eliminated from the first 16 numbers, the remaining numbers can be arranged into a *triangle equivalence* as shown.

$1 + 3 = 4$ $2 + 5 + 8 = 6 + 9$ $7 + 11 + 12 + 14 = 13 + 15 + 16$

Day 2

5 Given that a and b are real numbers such that for infinitely many positive integers m and n,

$$\lfloor an + b \rfloor \geq \lfloor a + bn \rfloor$$

$$\lfloor a + bm \rfloor \geq \lfloor am + b \rfloor$$

Prove that $a = b$.

6 Given a circle with center O, such that A is not on the circumcircle.
Let B be the reflection of A with respect to O. Now let P be a point on the circumcircle. The line perpendicular to AP through P intersects the circle at Q.
Prove that $AP \times BQ$ remains constant as P varies.

7 Determine all solutions of

$$x + y^2 = p^m$$

$$x^2 + y = p^n$$

For x, y, m, n positive integers and p being a prime.

8 Let $n > 1$ be a positive integer and $a_1, a_2, \ldots, a_{2n} \in \{-n, -n+1, \ldots, n-1, n\}$. Suppose

$$a_1 + a_2 + a_3 + \cdots + a_{2n} = n + 1$$

Prove that some of $a_1, a_2, \ldots, a_{2n}$ have sum 0.