RMM 2017 Shortlist

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by parmenides51, GGPiku, freghyy

–	Algebra
A1	A set A is endowed with a binary operation $*$ satisfying the following four conditions: (1) If a, b, c are elements of A, then $a * (b * c) = (a * b) * c$, (2) If a, b, c are elements of A such that $a * c = b * c$, then $a = b$, (3) There exists an element e of A such that $a * e = a$ for all a in A, and (4) If a and b are distinct elements of $A - \{e\}$, then $a^3 * b = b^3 * a^2$, where $x^k = x * x^{k-1}$ for all integers $k \geq 2$ and all x in A. Determine the largest cardinality A may have. proposed by Bojan Basic, Serbia
–	Combinatorics
C1	A planar country has an odd number of cities separated by pairwise distinct distances. Some of these cities are connected by direct two-way flights. Each city is directly connected to exactly two ther cities, and the latter are located farthest from it. Prove that, using these flights, one may go from any city to any other city
C2	Fix an integer $n \geq 2$ and let A be an $n \times n$ array with n cells cut out so that exactly one cell is removed out of every row and every column. A *stick* is a $1 \times k$ or $k \times 1$ subarray of A, where k is a suitable positive integer. (a) Determine the minimal number of *sticks* A can be dissected into. (b) Show that the number of ways to dissect A into a minimal number of *sticks* does not exceed 100^n. proposed by Palmer Mebane and Nikolai Beluhov a variation of part a, was problem 5 (`https://artofproblemsolving.com/community/c6h1389637p774307` a variation of part b, was posted here (`https://artofproblemsolving.com/community/c6h1389663p7743` this post was made in order to complete the post collection of RMM Shortlist 2017
–	Geometry
G1	Let $ABCD$ be a trapezium, $AD \parallel BC$, and let E, F be points on the sidesAB and CD, respectively. The circumcircle of AEF meets AD again at A_1, and the circumcircle of CEF meets BC again at C_1. Prove that A_1C_1, BD, EF are concurrent.

G2 Let ABC be a triangle. Consider the circle ω_B internally tangent to the sides BC and BA, and to the circumcircle of the triangle ABC, let P be the point of contact of the two circles. Similarly, consider the circle ω_C internally tangent to the sides CB and CA, and to the circumcircle of the triangle ABC, let Q be the point of contact of the two circles. Show that the incentre of the triangle ABC lies on the segment PQ if and only if $AB+AC=3BC$.

proposed by Luis Eduardo Garcia Hernandez, Mexico

G3 Let $ABCD$ be a convex quadrilateral and let P and Q be variable points inside this quadrilateral so that $\angle APB = \angle CPD = \angle AQB = \angle CQD$. Prove that the lines PQ obtained in this way all pass through a fixed point , or they are all parallel.

– Number Theory

N1 For each positive integer k, let $S(k)$ the sum of digits of k in decimal system.
Show that there is an integer k, with no 9 in it's decimal representation, such that:

$$S(2^{24^{2017}}k) = S(k)$$

N2 Let x, y and k be three positive integers. Prove that there exist a positive integer N and a set of $k+1$ positive integers $\{b_0, b_1, b_2, ..., b_k\}$, such that, for every $i = 0, 1, ..., k$, the b_i-ary expansion of N is a 3-digit palindrome, and the b_0-ary expansion is exactly $\overline{\text{xyx}}$.

proposed by Bojan Basic, Serbia
