Final Round - Korea 2015

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– Day 1, March 21st

1 Find all functions $f:R\rightarrow R$ such that
$f(x^{2015}+(f(y))^{2015})=(f(x))^{2015}+y^{2015}$ holds for all reals x,y

2 In a triangle $\triangle ABC$ with incenter I, the incircle meets lines BC,CA,AB at D,E,F respectively.
Define the circumcenter of $\triangle IAB$ and $\triangle IAC$ O_1 and O_2 respectively.
Let the two intersections of the circumcircle of $\triangle ABC$ and line EF be P,Q.
Prove that the circumcenter of $\triangle DPQ$ lies on the line O_1O_2.

3 There are at least 3 subway stations in a city.
In this city, there exists a route that passes through more than L subway stations, without revisiting.
Subways run both ways, which means that if you can go from subway station A to B, you can also go from B to A.
Prove that at least one of the two holds.

(i). There exists three subway stations A, B, C such that there does not exist a route from A to B which doesn't pass through C.

(ii). There is a cycle passing through at least $\lfloor \sqrt{2L} \rfloor$ stations, without revisiting a same station more than once.

– Day 2, March 22nd

4 $\triangle ABC$ is an acute triangle and its orthocenter is H.
The circumcircle of $\triangle ABH$ intersects line BC at D.
Lines DH and AC meets at P, and the circumcenter of $\triangle ADP$ is Q.
Prove that the circumcenter of $\triangle ABH$ lies on the circumcircle of $\triangle BDQ$.

5 For a fixed positive integer k, there are two sequences A_n and B_n.
They are defined inductively, by the following recurrences. $A_1=k$, $A_2=k$, $A_{n+2}=A_nA_{n+1}$
$B_1=1$, $B_2=k$, $B_{n+2}=\frac{B^3_{n+1}+1}{B_n}$
Prove that for all positive integers n, $A_{2n}B_{n+3}$ is an integer.

6 There are 2015 distinct circles in a plane, with radius 1.
Prove that you can select 27 circles, which form a set C, which satisfy the following.

For two arbitrary circles in C, they intersect with each other or
For two arbitrary circles in C, they don't intersect with each other.