JBMO Shortlist 2018

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– Algebra

A1 Let x,y,z be positive real numbers . Prove: $\frac{x}{\sqrt{\sqrt[4]{y}+\sqrt[4]{z}}}+\frac{y}{\sqrt{\sqrt[4]{z}+\sqrt[4]{x}}}+\frac{z}{\sqrt{\sqrt[4]{x}+\sqrt[4]{y}}}\geq\frac{\sqrt[4]{(\sqrt{x}+\sqrt{y}+\sqrt{z})^7}}{\sqrt{2\sqrt{27}}}$

A2 Find the maximum positive integer k such that for any positive integers m,n such that $m^3+n^3>(m+n)^2$, we have

$$m^3+n^3\geq (m+n)^2+k$$

Proposed by Dorlir Ahmeti, Albania

A3 Let a,b,c be positive real numbers . Prove that

$$\frac{1}{ab(b+1)(c+1)}+\frac{1}{bc(c+1)(a+1)}+\frac{1}{ca(a+1)(b+1)}\geq\frac{3}{(1+abc)^2}.$$

A4 Let $k>1, n>2018$ be positive integers, and let n be odd. The nonzero rational numbers $x_1,x_2,\ldots,x_n$ are not all equal and satisfy

$$x_1+\frac{k}{x_2}=x_2+\frac{k}{x_3}=x_3+\frac{k}{x_4}=\ldots=x_{n-1}+\frac{k}{x_n}=x_n+\frac{k}{x_1}$$

Find:
a) the product $x_1x_2\ldots x_n$ as a function of k and n
b) the least value of k, such that there exist $n,x_1,x_2,\ldots,x_n$ satisfying the given conditions.

A5 Let a, b,c,d and x,y,z,t be real numbers such that $0\leq a,b,c,d\leq 1$, $x,y,z,t\geq 1$ and $a+b+c+d+x+y+z+t=8$.
Prove that $a^2+b^2+c^2+d^2+x^2+y^2+z^2+t^2\leq 28$

A6 For a,b,c positive real numbers such that $ab+bc+ca=3$, prove: $\frac{a}{\sqrt{a^3+5}}+\frac{b}{\sqrt{b^3+5}}+\frac{c}{\sqrt{c^3+5}}\leq\frac{\sqrt{6}}{2}$

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A7 Let A be a set of positive integers satisfying the following :

$a.)$ If $n \in A$, then $n \leq 2018$.

$b.)$ If $S \subset A$ such that $|S| = 3$, then there exists $m, n \in S$ such that $|n - m| \geq \sqrt{n} + \sqrt{m}$

What is the maximum cardinality of A ?

– Combinatorics

C1 A set S is called *neighbouring* if it has the following two properties:
a) S has exactly four elements
b) for every element x of S, at least one of the numbers $x - 1$ or $x + 1$ belongs to S.
Find the number of all *neighbouring* subsets of the set $\{1, 2, ..., n\}$.

C2 Find max number n of numbers of three digits such that :
1. Each has digit sum 9
2. No one contains digit 0
3. Each 2 have different unit digits
4. Each 2 have different decimal digits
5. Each 2 have different hundreds digits

C3 The cells of a 8×8 table are initially white. Alice and Bob play a game. First Alice paints n of the fields in red. Then Bob chooses 4 rows and 4 columns from the table and paints all fields in them in black. Alice wins if there is at least one red field left. Find the least value of n such that Alice can win the game no matter how Bob plays.

– Geometry

G1 Let H be the orthocentre of an acute triangle ABC with $BC > AC$, inscribed in a circle Γ. The circle with centre C and radius CB intersects Γ at the point D, which is on the arc AB not containing C. The circle with centre C and radius CA intersects the segment CD at the point K. The line parallel to BD through K, intersects AB at point L. If M is the midpoint of AB and N is the foot of the perpendicular from H to CL, prove that the line MN bisects the segment CH.

G2 Let ABC be a right angled triangle with $\angle A = 90^o$ and AD its altitude. We draw parallel lines from D to the vertical sides of the triangle and we call E, Z their points of intersection with AB and AC respectively. The parallel line from C to EZ intersects the line AB at the point N. Let A' be the symmetric of A with respect to the line EZ and I, K the projections of A' onto AB and AC respectively. If T is the point of intersection of the lines IK and DE, prove that $\angle NA'T = \angle ADT$.

G3 Let $\triangle ABC$ and A',B',C' the symmetrics of vertex over opposite sides.The intersection of the circumcircles of $\triangle ABB'$ and $\triangle ACC'$ is A_1.B_1 and C_1 are defined similarly.Prove that lines

AA_1,BB_1 and CC_1 are concurent.

G4 Let ABC be a triangle with side-lengths a, b, c, inscribed in a circle with radius R and let I be ir's incenter. Let P_1, P_2 and P_3 be the areas of the triangles ABI, BCI and CAI, respectively. Prove that

$$\frac{R^4}{P_1^2}+\frac{R^4}{P_2^2}+\frac{R^4}{P_3^2}\geq 16$$

G5 Given a rectangle $ABCD$ such that $AB = b > 2a = BC$, let E be the midpoint of AD. On a line parallel to AB through point E, a point G is chosen such that the area of GCE is

$$(GCE)=\frac{1}{2}\left(\frac{a^3}{b}+ab\right)$$

Point H is the foot of the perpendicular from E to GD and a point I is taken on the diagonal AC such that the triangles ACE and AEI are similar. The lines BH and IE intersect at K and the lines CA and EH intersect at J. Prove that $KJ \perp AB$.

G6 Let XY be a chord of a circle Ω, with center O, which is not a diameter. Let P, Q be two distinct points inside the segment XY, where Q lies between P and X. Let ℓ the perpendicular line drawn from P to the diameter which passes through Q. Let M be the intersection point of ℓ and Ω, which is closer to P. Prove that

$$MP \cdot XY \geq 2 \cdot QX \cdot PY$$

– Number Theory

NT1 Find all integers m and n such that the fifth power of m minus the fifth power of n is equal to $16mn$.

NT2 Find all ordered pairs of positive integers (m, n) such that : $125 * 2^n - 3^m = 271$

NT3 Find all four-digit positive integers $\overline{abcd} = 10^3a + 10^2b + 10c + d$ $(a \neq 0)$ such that: $\overline{abcd} = a^{a+b+c+d} - a^{-a+b-c+d} + a$

NT4 Prove that there exist infinitely many positive integers n such that $\frac{4^n+2^n+1}{n^2+n+1}$ is a positive integer.