Finals 2019

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–	Day 1
1	Let ABC be an acute triangle. Points X and Y lie on the segments AB and AC, respectively, such that $AX = AY$ and the segment XY passes through the orthocenter of the triangle ABC. Lines tangent to the circumcircle of the triangle AXY at points X and Y intersect at point P. Prove that points A, B, C, P are concyclic.
2	Let p a prime number and r an integer such that $p \| r^7 - 1$. Prove that if there exist integers a, b such that $p \| r + 1 - a^2$ and $p \| r^2 + 1 - b^2$, then there exist an integer c such that $p \| r^3 + 1 - c^2$.
3	$n \geq 3$ guests met at a party. Some of them know each other but there is no quartet of different guests a, b, c, d such that in pairs $\{a, b\}, \{b, c\}, \{c, d\}, \{d, a\}$ guests know each other but in pairs $\{a, c\}, \{b, d\}$ guests don't know each other. We say a nonempty set of guests X is an *ingroup*, when guests from X know each other pairwise and there are no guests not from X knowing all guests from X. Prove that there are at most $\frac{n(n-1)}{2}$ different ingroups at that party.
–	Day 2
4	Let n, k, ℓ be positive integers and $\sigma : \{1, 2, \ldots, n\} \rightarrow \{1, 2, \ldots, n\}$ an injection such that $\sigma(x) - x \in \{k, -\ell\}$ for all $x \in \{1, 2, \ldots, n\}$. Prove that $k + \ell \| n$.

5 The sequence $a_1, a_2, \ldots, a_n$ of positive real numbers satisfies the following conditions:

$$\sum_{i=1}^{n} \frac{1}{a_i} \leq 1 \quad \text{and} \quad a_i \leq a_{i-1} + 1$$

for all $i \in \{1, 2, \ldots, n\}$, where a_0 is an integer. Prove that

$$n \leq 4a_0 \cdot \sum_{i=1}^{n} \frac{1}{a_i}$$

6 Denote by Ω the circumcircle of the acute triangle ABC. Point D is the midpoint of the arc BC of Ω not containing A. Circle ω centered at D is tangent to the segment BC at point E. Tangents to the circle ω passing through point A intersect line BC at points K and L such that points B, K, L, C lie on the line BC in that order. Circle γ_1 is tangent to the segments AL and BL and to the circle Ω at point M. Circle γ_2 is tangent to the segments AK and CK and to the circle Ω at point N. Lines KN and LM intersect at point P. Prove that $\sphericalangle KAP = \sphericalangle EAL$.
