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304 Two equal chess-boards (8×8) have the same centre, but one is rotated by 45 degrees with respect to another. Find the total area of black fields intersection, if the fields have unit length sides.

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305 Given points A, B, M, N on the circumference. Two chords $[MA_1]$ and $[MA_2]$ are orthogonal to (NA) and (NB) lines respectively. Prove that (AA_1) and (BB_1) lines are parallel.

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306 Let us say, that a natural number has the property $P(k)$ if it can be represented as a product of k succeeding natural numbers greater than 1.
a) Find k such that there exists n which has properties $P(k)$ and $P(k+2)$ simultaneously.
b) Prove that there is no number having properties $P(2)$ and $P(4)$ simultaneously

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307 The rectangular table has four rows.
The first one contains arbitrary natural numbers (some of them may be equal). The consecutive lines are filled according to the rule: we look through the previous row from left to the certain number n and write the number k if n was met k times.
Prove that the second row coincides with the fourth one.

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308 Given real a.
Find the least possible area of the rectangle with the sides parallel to the coordinate axes and containing the figure determined by the system of inequalities $y \le -x^2$ and $y \ge x^2 - 2x + a$

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309 Three equilateral triangles ABC, CDE, EHK (the vertices are mentioned counterclockwise) are lying in the plane so, that the vectors $\overrightarrow{AD}$ and $\overrightarrow{DK}$ are equal. Prove that the triangle BHD is also equilateral

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310 There are 1000 inhabitants in a settlement.
Every evening every inhabitant tells all his friends all the news he had heard the previous day.
Every news becomes finally known to every inhabitant.
Prove that it is possible to choose 90 of inhabitants so, that if you tell them a news simultaneously, it will be known to everybody in 10 days.

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311 It is known about real a and b that the inequality $a \cos x + b \cos(3x) > 1$ has no real solutions.
Prove that $|b| \le 1$.

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312 The points K and M are the centres of the AB and CD sides of the convex quadrangle $ABCD$.
The points L and M belong to two other sides and $KLMN$ is a rectangle.

Prove that $KLMN$ area is a half of $ABCD$ area.

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313 Find all the sequences of natural k_n with two properties:
a) $k_n \le n\sqrt{n}$ for all n
b) $(k_n - k_m)$ is divisible by $(m - n)$ for all $m > n$

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314 Is it possible to fill a rectangular table with black and white squares (only) so, that the number of black squares will equal to the number of white squares, and each row and each column will have more than 75% squares of the same colour?

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315 The quadrangles $AMBE$, $AHBT$, $BKXM$, and $CKXP$ are parallelograms.
Prove that the quadrangle $ABTE$ is also parallelogram.
(the vertices are mentioned counterclockwise)

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316 Find the natural solutions of the equation $x^3 - y^3 = xy + 61$.

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317 Eighteen soccer teams have played 8 tours of a one-round tournament.
Prove that there is a triple of teams, having not met each other yet.

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318 The points C_1, A_1, B_1 belong to $[AB], [BC], [CA]$ sides, respectively, of the ABC triangle. $\frac{|AC_1|}{|C_1B|} = \frac{|BA_1|}{|A_1C|} = \frac{|CB_1|}{|B_1A|} = \frac{1}{3}$.
Prove that the perimeter P of the ABC triangle and the perimeter p of the $A_1B_1C_1$ triangle, satisfy inequality $\frac{P}{2} < p < \frac{3P}{4}$.

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319 Positive numbers x, y satisfy equality $x^3 + y^3 = x - y$.
Prove that $x^2 + y^2 < 1$.

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320 A pupil has tried to make a copy of a convex polygon, drawn inside the unit circle. He draw one side, from its end – another, and so on. Having finished, he has noticed that the first and the last vertices do not coincide, but are situated d units of length far from each other. The pupil draw angles precisely, but made relative error less than p in the lengths of sides. Prove that $d < 4p$.

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321 A number is written in the each vertex of a cube. It is allowed to add one to two numbers written in the ends of one edge. Is it possible to obtain the cube with all equal numbers if the numbers were initially as on the pictures:

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322 Find n such that each of the numbers $n, (n + 1), ..., (n + 20)$ has the common divider greater than one with the number $30030 = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13$.

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323 The natural numbers from 100 to 999 are written on separate cards. They are gathered in one pile with their numbers down in arbitrary order. Let us open them in sequence and divide into

10 piles according to the least significant digit. The first pile will contain cards with 0 at the end, ... , the tenth – with 9. Then we shall gather 10 piles in one pile, the first – down, then the second, ... and the tenth – up. Let us repeat the procedure twice more, but the next time we shall divide cards according to the second digit, and the last time – to the most significant one.
What will be the order of the cards in the obtained pile?

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324 Six points are marked inside the $3x4$ rectangle.
Prove that there is a pair of marked points with the distance between them not greater than $\sqrt{5}$.

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325 a) Find the minimal value of the polynomial $P(x, y) = 4 + x^2y^4 + x^4y^2 - 3x^2y^2$
b) Prove that it cannot be represented as a sum of the squares of some polynomials of x, y.

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326 The segments $[AD]$, $[BE]$ and $[CF]$ are the side edges of the right triangle prism. (the equilateral triangle is a base)
Find all the points in its base ABC, situated on the equal distances from the (AE), (BF) and (CD) lines.
