All Soviet Union Mathematical Olympiad 1984

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371 a) The product of n integers equals n, and their sum is zero. Prove that n is divisible by 4.
b) Let n is divisible by 4.
Prove that there exist n integers such, that their product equals n, and their sum is zero.

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372 Prove that every positive a and b satisfy inequality $\frac{(a+b)^2}{2}+\frac{a+b}{4}\ge a\sqrt{b}+b\sqrt{a}$

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373 Given two equilateral triangles $A_1B_1C_1$ and $A_2B_2C_2$ in the plane. (The vertices are mentioned counterclockwise.)
We draw vectors $\overrightarrow{OA},\overrightarrow{OB},\overrightarrow{OC}$, from the arbitrary point O, equal to $\overrightarrow{A_1A_2},\overrightarrow{B_1B_2},\overrightarrow{C_1C_2}$ respectively.
Prove that the triangle ABC is equilateral.

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374 Given four colours and enough square plates 1×1. We have to paint four edges of every plate with four different colours and combine plates, putting them with the edges of the same colour together.
Describe all the pairs m,n, such that we can combine those plates in a $n\times m$ rectangle, that has every edge of one colour, and its four edges have different colours.

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375 Prove that every positive x,y and real a satisfy inequality $x^{\sin^2 a}y^{\cos^2 a}<x+y$.

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376 Given a cube and two colours. Two players paint in turn a triple of arbitrary unpainted edges with his colour. (Everyone makes two moves.) The first wins if he has painted all the edges of some face with his colour. Can he always win?

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377 n natural numbers ($n>3$) are written on the circumference. The relation of the two neighbours sum to the number itself is a whole number. Prove that the sum of those relations is
a) not less than $2n$
b) less than $3n$

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378 The circle with the centre O is inscribed in the ABC triangle. The circumference touches its sides $[BC],[CA],[AB]$ in A_1,B_1,C_1 points respectively. The $[AO],[BO],[CO]$ segments cross the circumference in A_2,B_2,C_2 points respectively. Prove that $(A_1A_2),(B_1B_2)$ and (C_1C_2) lines intersect in one point.

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379 Find integers m and n such that $(5+3\sqrt{2})^m=(3+5\sqrt{2})^n$.

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380 n real numbers are written in increasing order in a line. The same numbers are written in the second line below in unknown order. The third line contains the sums of the pairs of numbers above from two previous lines. It comes out, that the third line is arranged in increasing order. Prove that the second line coincides with the first one.

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381 Given ABC triangle. From the P point three lines $(PA),(PB),(PC)$ are drawn. They cross the circumscribed circle at A_1,B_1,C_1 points respectively. It comes out that the $A_1B_1C_1$ triangle equals to the initial one. Prove that there are not more than eight such a points P in a plane.

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382 Positive x,y,z satisfy a system: $\begin{cases} x^2+xy+y^2/3=25 \\ y^2/3+z^2=9 \\ z^2+zx+x^2=16 \end{cases}$

Find the value of expression $xy+2yz+3zx$.

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383 The teacher wrote on a blackboard: " $x^2+10x+20$ "
Then all the pupils in the class came up in turn and either decreased or increased by 1 either the free coefficient or the coefficient at x, but not both. Finally they have obtained: " $x^2+20x+10$ ". Is it true that some time during the process there was written the square polynomial with the integer roots?

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384 The centre of the coin with radius r is moved along some polygon with the perimeter P, that is circumscribed around the circle with radius R ($R > r$). Find the coin trace area (a sort of polygon ring).

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385 There are scales and $(n+1)$ weights with the total weight $2n$. Each weight is an integer. We put all the weights in turn on the lighter side of the scales, starting from the heaviest one, and if the scales is in equilibrium – on the left side.
Prove that when all the weights will be put on the scales, they will be in equilibrium.

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386 Let us call "absolutely prime" the prime number, if having transposed its digits in an arbitrary order, we obtain prime number again. Prove that its notation cannot contain more than three different digits.

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387 The x and y figures satisfy a condition:
for every $n\ge 1$ the number $xx...x6yy...y4$ (n times x and n times y) is an exact square.
Find all possible x and y.

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388 The A,B,C and D points (from left to right) belong to the straight line.
Prove that every point E, that doesn't belong to the line satisfy: $|AE|+|ED|+||AB|-|CD||>|BE|+|CE|$.

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389 Given a sequence $\{x_n\}, x_1 = x_2 = 1, x_{n+2} = x_{n+1}^2 - \frac{x_n}{2}$.
Prove that the sequence has limit and find it.

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390 The white fields of $1983 \times 1984$1983x1984 are filled with either $+1$ or -1.
For every black field, the product of neighbouring numbers is $+1$.
Prove that all the numbers are $+1$.

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392 What is more $\frac{2}{201}$ or $\ln \frac{101}{100}$? (No differential calculus allowed).

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393 Given three circles c_1, c_2, c_3 with r_1, r_2, r_3 radiuses, $r_1 > r2, r_1 > r_3$. Each lies outside of two others. The A point – an intersection of the outer common tangents to c_1 and c_2 – is outside c_3. The B point – an intersection of the outer common tangents to c_1 and c_3 – is outside c_2. Two pairs of tangents – from A to c_3 and from B to c_2 – are drawn.
Prove that the quadrangle, they make, is circumscribed around some circle and find its radius.

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394 Prove that every cube's cross-section, containing its centre, has the area not less then its face's area.

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391 The white fields of $3x3$ chess-board are filled with either $+1$ or -1.
For every field, let us calculate the product of neighbouring numbers.
Then let us change all the numbers by the respective products.
Prove that we shall obtain only $+1$'s, having repeated this operation finite number of times.
