All Soviet Union Mathematical Olympiad 1985

www.artofproblemsolving.com/community/c907021

by parmenides51

395 Two perpendiculars are drawn from the middles of each side of the acute-angle triangle to two other sides. Those six segments make hexagon. Prove that the hexagon area is a half of the triangle area.

---

396 Is there any numbber n, such that the sum of its digits in the decimal notation is 1000, and the sum of its square digits in the decimal notation is 1000000?

---

397 What maximal number of the men in checkers game can be put on the chess-board 8×8 so, that every man can be taken by at least one other man ?

---

398 You should paint all the sides and diagonals of the right n-angle so, that every pair of segments, having the common point, would be painted with different colours. How many colours will you require?

---

399 Given a straight line ℓ and the point O out of the line.
Prove that it is possible to move an arbitrary point A in the same plane to the O point, using only rotations around O and symmetry with respect to the ℓ.

---

400 The senior coefficient a in the square polynomial $P(x) = ax^2 + bx + c$ is more than 100.
What is the maximal number of integer values of x, such that $|P(x)| < 50$.

---

401 In the diagram below $a, b, c, d, e, f, g, h, i, j$ are distinct positive integers and each (except a, e, h and j) is the sum of the two numbers to the left and above. For example, $b = a+e, f = e+h, i = h + j$. What is the smallest possible value of d?
j
h i
e f g
a b c d

---

402 Given unbounded strictly increasing sequence $a_1, a_2, ..., a_n, ...$ of positive numbers. Prove that
a) there exists a number k_0 such that for all $k > k_0$ the following inequality is valid: $\frac{a_1}{a_2} + \frac{a_2}{a_3} + ... + \frac{a_k}{a_{k-1}} < k - 1$
b) there exists a number k_0 such that for all $k > k_0$ the following inequality is valid: $\frac{a_1}{a_2} + \frac{a_2}{a_3} + ... + \frac{a_k}{a_{k-1}} < k - 1985$

---

403 Find all the pairs (x, y) such that $|\sin x - \sin y| + \sin x \sin y \leq 0$.

---

404 The convex pentagon $ABCDE$ was drawn in the plane. A_1 was symmetric to A with respect to B. B_1 was symmetric to B with respect to C. C_1 was symmetric to C with respect to D. D_1 was symmetric to D with respect to E. E_1 was symmetric to E with respect to A.
How is it possible to restore the initial pentagon with the compasses and ruler, knowing A_1, B_1, C_1, D_1, E_1 points?

---

405 The sequence $a_1, a_2, ..., a_k, ...$ is built according to the rules: $a_{2n} = a_n, a_{4n+1} = 1, a_{4n+3} = 0$.
Prove that it is non-periodical sequence.

---

406 n straight lines are drawn in a plane.
They divide the plane onto several parts. Some of the parts are painted.
Not a pair of painted parts has non-zero length common bound.
Prove that the number of painted parts is not more than $\frac{n^2+n}{3}$.

---

407 Given a cube, a cubic box, that exactly suits for the cube, and six colours.
First man paints each side of the cube with its (side's) unique colour. Another man does the same with the box.
Prove that the third man can put the cube in the box in such a way, that every cube side will touch the box side of different colour.

---

408 The $[A_0A_5]$ diameter divides a circumference with the O centre onto two hemicircumferences. One of them is divided onto five equal arcs $A_0A_1, A_1A_2, A_2A_3, A_3A_4, A_4A_5$. The (A_1A_4) line crosses (OA_2) and (OA_3) lines in M and N points.
Prove that $(|A_2A_3| + |MN|)$ equals to the circumference radius.

---

409 If there are four numbers (a, b, c, d) in four registers of the calculating machine, they turn into $(a-b, b-c, c-d, d-a)$ numbers whenever you press the button. Prove that if not all the initial numbers are equal, machine will obtain at least one number more than 1985 after some number of the operations.

---

410 Numbers $1, 2, 3, ..., 2n$ are divided onto two equal groups. Let $a_1, a_2, ..., a_n$ be the first group numbers in the increasing order, and $b_1, b_2, ..., b_n$ – the second group numbers in the decreasing order.
Prove that $|a_1 - b_1| + |a_2 - b_2| + ... + |a_n - b_n| = n^2$.

---

411 The parallelepiped is constructed of the equal cubes. Three parallelepiped faces, having the common vertex are painted. Exactly half of all the cubes have at least one face painted. What is the total number of the cubes?

---

412 One of two circumferences of radius R comes through A and B vertices of the $ABCD$ parallelogram. Another comes through B and D. Let M be another point of circumferences intersection.
Prove that the circle circumscribed around AMD triangle has radius R.

---

413 Given right hexagon.
The lines parallel to all the sides are drawn from all the vertices and middles of the sides (consider only the interior, with respect to the hexagon, parts of those lines). Thus the hexagon is divided onto 24 triangles, and the figure has 19 nodes. 19 different numbers are written in those nodes.
Prove that at least 7 of 24 triangles have the property: the numbers in its vertices increase (from the least to the greatest) counterclockwise.

---

414 Solve the equation ("2" encounters 1985 times):

$$\frac{x}{2+\frac{x}{2+\frac{x}{2+\cdots\frac{x}{2+\sqrt{1+x}}}}}=1$$

---

415 All the points situated more close than 1 cm to ALL the vertices of the regular pentagon with 1 cm side, are deleted from that pentagon. Find the area of the remained figure.

---

416 Given big enough sheet of cross-lined paper with the side of the squares equal to 1. We are allowed to cut it along the lines only.
Prove that for every $m > 12$ we can cut out a rectangle of the greater than m area such, that it is impossible to cut out a rectangle of m area from it.

---

417 The $ABCDA_1B_1C_1D_1$ cube has unit length edges.
Find the distance between two circumferences, one of those is inscribed into the $ABCD$ base, and another comes through A, C and B_1 points.
