All Soviet Union Mathematical Olympiad 1988

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by parmenides51

463 A book contains 30 stories. Each story has a different number of pages under 31. The first story starts on page 1 and each story starts on a new page. What is the largest possible number of stories that can begin on odd page numbers?

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464 $ABCD$ is a convex quadrilateral. The midpoints of the diagonals and the midpoints of AB and CD form another convex quadrilateral Q. The midpoints of the diagonals and the midpoints of BC and CA form a third convex quadrilateral Q'. The areas of Q and Q' are equal. Show that either AC or BD divides $ABCD$ into two parts of equal area.

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465 Show that there are infinitely many triples of distinct positive integers a, b, c such that each divides the product of the other two and $a + b = c + 1$.

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466 Given a sequence of 19 positive integers not exceeding 88 and another sequence of 88 positive integers not exceeding 19. Show that we can find two subsequences of consecutive terms, one from each sequence, with the same sum.

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467 The quadrilateral $ABCD$ is inscribed in a fixed circle. It has AB parallel to CD and the length AC is fixed, but it is otherwise allowed to vary. If h is the distance between the midpoints of AC and BD and k is the distance between the midpoints of AB and CD, show that the ratio h/k remains constant.

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468 The numbers 1 and 2 are written on an empty blackboard.
Whenever the numbers m and n appear on the blackboard the number $m + n + mn$ may be written.
Can we obtain :
(1) 13121,
(2) 12131?

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469 If rationals x, y satisfy $x^5 + y^5 = 2x^2y^2$, show that $1 - xy$ is the square of a rational.

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470 There are 21 towns. Each airline runs direct flights between every pair of towns in a group of five. What is the minimum number of airlines needed to ensure that at least one airline runs direct flights between every pair of towns?

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471 Find all positive integers n satisfying $\left(1 + \frac{1}{n}\right)^{n+1} = \left(1 + \frac{1}{1998}\right)^{1998}$.

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472 A, B, C are the angles of a triangle. Show that $2\frac{\sin A}{A}+2\frac{\sin B}{B}+2\frac{\sin C}{C} \le \left(\frac{1}{B}+\frac{1}{C}\right)\sin A+\left(\frac{1}{C}+\frac{1}{A}\right)\sin B+\left(\frac{1}{A}+\frac{1}{B}\right)\sin C$

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473 Form $10A$ has 29 students who are listed in order on its duty roster. Form $10B$ has 32 students who are listed in order on its duty roster. Every day two students are on duty, one from form $10A$ and one from form $10B$. Each day just one of the students on duty changes and is replaced by the following student on the relevant roster (when the last student on a roster is replaced he is replaced by the first). On two particular days the same two students were on duty. Is it possible that starting on the first of these days and ending the day before the second, every pair of students (one from $10A$ and one from $10B$) shared duty exactly once?

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474 In the triangle ABC, the angle C is obtuse and D is a fixed point on the side BC, different from B and C. For any point M on the side BC, different from D, the ray AM intersects the circumcircle S of ABC at N. The circle through M, D and N meets S again at P, different from N. Find the location of the point M which minimises MP.

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475 Show that there are infinitely many odd composite numbers in the sequence $1^1, 1^1+2^2, 1^1+2^2+3^3, 1^1+2^2+3^3+4^4, ...$.

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476 ABC is an acute-angled triangle. The tangents to the circumcircle at A and C meet the tangent at B at M and N. The altitude from B meets AC at P. Show that BP bisects the angle MPN

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477 What is the minimal value of $\frac{b}{c+d}+\frac{c}{a+b}$ for positive real numbers b and c and non-negative real numbers a and d such that $b+c \ge a+d$?

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478 n^2 real numbers are written in a square $n \times n$ table so that the sum of the numbers in each row and column equals zero. A move is to add a row to one column and subtract it from another (so if the entries are a_{ij} and we select row i, column h and column k, then column h becomes $a_{1h}+a_{i1}, a_{2h}+a_{i2}, ..., a_{nh}+a_{in}$, column k becomes $a_{1k}-a_{i1}, a_{2k}-a_{i2}, ..., a_{nk}-a_{in}$, and the other entries are unchanged). Show that we can make all the entries zero by a series of moves.

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479 In the acute-angled triangle ABC, the altitudes BD and CE are drawn. Let F and G be the points of the line ED such that BF and CG are perpendicular to ED. Prove that $EF = DG$.

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480 Find the minimum value of $\frac{xy}{z}+\frac{yz}{x}+\frac{zx}{y}$ for positive reals x, y, z with $x^2+y^2+z^2=1$.

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481 A polygonal line connects two opposite vertices of a cube with side 2. Each segment of the line has length 3 and each vertex lies on the faces (or edges) of the cube. What is the smallest number of segments the line can have?

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482 Let m, n, k be positive integers with $m \ge n$ and $1+2+...+n = mk$. Prove that the numbers $1, 2, ..., n$ can be divided into k groups in such a way that the sum of the numbers in each group equals m.

483 A polygonal line with a finite number of segments has all its vertices on a parabola. Any two adjacent segments make equal angles with the tangent to the parabola at their point of intersection. One end of the polygonal line is also on the axis of the parabola. Show that the other vertices of the polygonal line are all on the same side of the axis.

484 What is the smallest n for which there is a solution to

$$\begin{cases}\sin x_1 + \sin x_2 + ... + \sin x_n = 0\\ \sin x_1 + 2\sin x_2 + ... + n\sin x_n = 100\end{cases}$$

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485 The sequence of integers an is given by $a_0 = 0, a_n = p(a_n - 1)$, where $p(x)$ is a polynomial whose coefficients are all positive integers. Show that for any two positive integers m, k with greatest common divisor d, the greatest common divisor of a_m and a_k is a_d.

486 Prove that for any tetrahedron the radius of the inscribed sphere $r < \frac{ab}{2(a+b)}$, where a and b are the lengths of any pair of opposite edges.