All Soviet Union Mathematical Olympiad 1989

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487 7 boys each went to a shop 3 times. Each pair met at the shop. Show that 3 must have been in the shop at the same time.

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488 Can 77 blocks each $3 \times 3 \times 1$ be assembled to form a $7 \times 9 \times 11$ block?

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489 The incircle of ABC touches AB at M. N is any point on the segment BC. Show that the incircles of AMN, BMN, ACN have a common tangent.

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490 A positive integer n has exactly 12 positive divisors $1 = d_1 < d_2 < d_3 < ... < d_{12} = n$. Let $m = d_4 - 1$. We have $d_m = (d_1 + d_2 + d_4)d_8$. Find n.

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491 Eight pawns are placed on a chessboard, so that there is one in each row and column. Show that an even number of the pawns are on black squares.

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492 ABC is a triangle. A', B', C' are points on the segments BC, CA, AB respectively. $\angle B'A'C' = \angle A$, $\frac{AC'}{C'B} = \frac{BA'}{A'C} = \frac{CB'}{B'A}$. Show that ABC and $A'B'C'$ are similar.

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493 One bird lives in each of n bird-nests in a forest. The birds change nests, so that after the change there is again one bird in each nest. Also for any birds A, B, C, D (not necessarily distinct), if the distance $AB < CD$ before the change, then $AB > CD$ after the change. Find all possible values of n.

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494 Show that the 120 five digit numbers which are permutations of 12345 can be divided into two sets with each set having the same sum of squares.

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495 We are given 1998 normal coins, 1 heavy coin and 1 light coin, which all look the same. We wish to determine whether the average weight of the two abnormal coins is less than, equal to, or greater than the weight of a normal coin. Show how to do this using a balance 4 times or less.

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496 A triangle with perimeter 1 has side lengths a, b, c. Show that $a^2 + b^2 + c^2 + 4abc < \frac{1}{2}$.

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497 $ABCD$ is a convex quadrilateral. X lies on the segment AB with $\frac{AX}{XB} = \frac{m}{n}$. Y lies on the segment CD with $\frac{CY}{YD} = \frac{m}{n}$. AY and DX intersect at P, and BY and CX intersect at Q. Show that $\frac{S_{XQYP}}{S_{ABCD}} < \frac{mn}{m^2+mn+n^2}$.

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498 A 23×23 square is tiled with $1 \times 1, 2 \times 2$ and 3×3 squares. What is the smallest possible number of 1×1 squares?

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499 Do there exist two reals whose sum is rational, but the sum of their n th powers is irrational for all $n > 1$?
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500 An insect is on a square ceiling side 1. The insect can jump to the midpoint of the segment joining it to any of the four corners of the ceiling. Show that in 8 jumps it can get to within $1/100$ of any chosen point on the ceiling

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501 $ABCD$ has $AB = CD$, but AB not parallel to CD, and AD parallel to BC. The triangle is ABC is rotated about C to $A'B'C$. Show that the midpoints of $BC, B'C$ and $A'D$ are collinear.

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502 Show that for each integer $n > 0$, there is a polygon with vertices at lattice points and all sides parallel to the axes, which can be dissected into 1×2 (and / or 2×1) rectangles in exactly n ways.

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503 Find the smallest positive integer n for which we can find an integer m such that $\left[\frac{10^n}{m}\right] = 1989$.

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504 ABC is a triangle. Points D, E, F are chosen on BC, CA, AB such that B is equidistant from D and F, and C is equidistant from D and E. Show that the circumcenter of AEF lies on the bisector of EDF.

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505 S and S' are two intersecting spheres. The line BXB' is parallel to the line of centers, where B is a point on S, B' is a point on S' and X lies on both spheres. A is another point on S, and A' is another point on S' such that the line AA' has a point on both spheres. Show that the segments AB and $A'B'$ have equal projections on the line AA'.

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506 Two walkers are at the same altitude in a range of mountains. The path joining them is piecewise linear with all its vertices above the two walkers. Can they each walk along the path until they have changed places, so that at all times their altitudes are equal?

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507 Find the least possible value of $(x + y)(y + z)$ for positive reals satisfying $(x + y + z)xyz = 1$.

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508 A polyhedron has an even number of edges. Show that we can place an arrow on each edge so that each vertex has an even number of arrows pointing towards it (on adjacent edges).

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509 N is the set of positive integers. Does there exist a function $f : N \to N$ such that $f(n + 1) = f(f(n)) + f(f(n + 2))$ for all n?

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510 A convex polygon is such that any segment dividing the polygon into two parts of equal area which has at least one end at a vertex has length < 1. Show that the area of the polygon is $< \pi/4$.