Czech-Polish-Slovak Match 2019

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– Day 1

1 Let ω be a circle. Points A, B, C, X, D, Y lie on ω in this order such that BD is its diameter and $DX = DY = DP$, where P is the intersection of AC and BD. Denote by E, F the intersections of line XP with lines AB, BC, respectively. Prove that points B, E, F, Y lie on a single circle.

2 We consider positive integers n having at least six positive divisors. Let the positive divisors of n be arranged in a sequence $(d_i)_{1\le i\le k}$ with

$$1 = d_1 < d_2 < \cdots < d_k = n \quad (k \ge 6).$$

Find all positive integers n such that

$$n = d_5^2 + d_6^2.$$

3 A dissection of a convex polygon into finitely many triangles by segments is called a *trilateration* if no three vertices of the created triangles lie on a single line (vertices of some triangles might lie inside the polygon). We say that a trilateration is *good* if its segments can be replaced with one-way arrows in such a way that the arrows along every triangle of the trilateration form a cycle and the arrows along the whole convex polygon also form a cycle. Find all $n \ge 3$ such that the regular n-gon has a good trilateration.

– Day 2

4 Given a real number α, find all pairs (f, g) of functions $f, g : \mathbb{R} \to \mathbb{R}$ such that

$$xf(x + y) + \alpha \cdot yf(x - y) = g(x) + g(y) \qquad , \forall x, y \in \mathbb{R}.$$

5 Determine whether there exist 100 disks $D_2, D_3, \ldots, D_{101}$ in the plane such that the following conditions hold for all pairs (a, b) of indices satisfying $2 \le a < b \le 101$:

- If $a|b$ then D_a is contained in D_b.
- If $\gcd(a, b) = 1$ then D_a and D_b are disjoint.

(A disk $D(O, r)$ is a set of points in the plane whose distance to a given point O is at most a given positive real number r.)

6 Let ABC be an acute triangle with $AB < AC$ and $\angle BAC = 60^\circ$. Denote its altitudes by AD, BE, CF and its orthocenter by H. Let K, L, M be the midpoints of sides BC, CA, AB, respectively. Prove that the midpoints of segments AH, DK, EL, FM lie on a single circle.