Silk Road Mathematics Competiton 2019

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by parmenides51

1 The altitudes of the acute-angled non-isosceles triangle ABC intersect at the point H. On the segment C_1H, where CC_1 is the altitude of the triangle, the point K is marked. Points L and M are the feet of perpendiculars from point K on straight lines AC and BC, respectively. The lines AM and BL intersect at N. Prove that $\angle ANK = \angle HNL$.

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2 Let $a_1, a_2, \dots, a_{99}$ be positive real numbers such that $ia_j + ja_i \ge i + j$ for all $1 \le i < j \le 99.$ Prove , that $(a_1 + 1)(a_2 + 2) \dots (a_{99} + 99) \ge 100!$.

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3 Find all pairs of (a, n) natural numbers such that $\varphi(a^n + n) = 2^n$.
($\varphi(n)$ is the Euler function, that is, the number of integers from 1 up to n, relative prime to n)

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4 The sequence $\{a_n\}$ is defined as follows: $a_0 = 1$ and $a_n = \sum_{k=1}^{[\sqrt{n}]} a_{n-k^2}$ for $n \ge 1.$
Prove that among $a_1, a_2, \dots, a_{10^6}$ there are at least 500 even numbers.
(Here, $[x]$ is the largest integer not exceeding x.)
