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India International Mathematics Olympiad Training Camp 2019

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– Day 1

P1 In an acute angled triangle ABC with $AB < AC$, let I denote the incenter and M the midpoint of side BC. The line through A perpendicular to AI intersects the tangent from M to the incircle (different from line BC) at a point P¿ Show that AI is tangent to the circumcircle of triangle MIP.

Proposed by Tejaswi Navilarekallu

P2 Show that there do not exist natural numbers $a_1, a_2, \ldots, a_{2018}$ such that the numbers

$$(a_1)^{2018} + a_2, (a_2)^{2018} + a_3, \ldots, (a_{2018})^{2018} + a_1$$

are all powers of 5

Proposed by Tejaswi Navilarekallu

P3 Let n be a given positive integer. Sisyphus performs a sequence of turns on a board consisting of $n+1$ squares in a row, numbered 0 to n from left to right. Initially, n stones are put into square 0, and the other squares are empty. At every turn, Sisyphus chooses any nonempty square, say with k stones, takes one of these stones and moves it to the right by at most k squares (the stone should say within the board). Sisyphus' aim is to move all n stones to square n.
Prove that Sisyphus cannot reach the aim in less than

$$\left\lceil\frac{n}{1}\right\rceil + \left\lceil\frac{n}{2}\right\rceil + \left\lceil\frac{n}{3}\right\rceil + \cdots + \left\lceil\frac{n}{n}\right\rceil$$

turns. (As usual, $\lceil x \rceil$ stands for the least integer not smaller than x.)

– Day 2

P1 Let $\mathbb{Q}_{>0}$ denote the set of all positive rational numbers. Determine all functions $f : \mathbb{Q}_{>0} \to \mathbb{Q}_{>0}$ satisfying

$$f(x^2 f(y)^2) = f(x)^2 f(y)$$

for all $x, y \in \mathbb{Q}_{>0}$

P2 Let n be a natural number. A tiling of a $2n \times 2n$ board is a placing of $2n^2$ dominos (of size 2×1 or 1×2) such that each of them covers exactly two squares of the board and they cover all the board.Consider now two *sepearate tilings* of a $2n \times 2n$ board: one with red dominos and the other with blue dominos. We say two squares are red neighbours if they are covered by the same red domino in the red tiling; similarly define blue neighbours.

Suppose we can assign a non-zero integer to each of the squares such that the number on any square equals the difference between the numbers on it's red and blue neighbours i.e the number on it's red neigbhbour minus the number on its blue neighbour. Show that n is divisible by 3

Proposed by Tejaswi Navilarekallu

P3 Let $f : \{1, 2, 3, \dots\} \to \{2, 3, \dots\}$ be a function such that $f(m + n)|f(m) + f(n)$ for all pairs m, n of positive integers. Prove that there exists a positive integer $c > 1$ which divides all values of f.

– Day 3

P1 Given any set S of positive integers, show that at least one of the following two assertions holds:

(1) There exist distinct finite subsets F and G of S such that $\sum_{x\in F} 1/x = \sum_{x\in G} 1/x$;

(2) There exists a positive rational number $r < 1$ such that $\sum_{x\in F} 1/x \neq r$ for all finite subsets F of S.

P2 Let ABC be an acute-angled scalene triangle with circumcircle Γ and circumcenter O. Suppose $AB < AC$. Let H be the orthocenter and I be the incenter of triangle ABC. Let F be the midpoint of the arc BC of the circumcircle of triangle BHC, containing H.

Let X be a point on the arc AB of Γ not containing C, such that $\angle AXH = \angle AFH$. Let K be the circumcenter of triangle XIA. Prove that the lines AO and KI meet on Γ.

Proposed by Anant Mudgal

P3 Let k be a positive integer. The organising commitee of a tennis tournament is to schedule the matches for $2k$ players so that every two players play once, each day exactly one match is played, and each player arrives to the tournament site the day of his first match, and departs the day of his last match. For every day a player is present on the tournament, the committee has to pay 1 coin to the hotel. The organisers want to design the schedule so as to minimise the total cost of all players' stays. Determine this minimum cost.

– Day 4

P1 Determine all non-constant monic polynomials $f(x)$ with integer coefficients for which there exists a natural number M such that for all $n \geq M$, $f(n)$ divides $f(2^n) - 2^{f(n)}$

Proposed by Anant Mudgal

P2 Determine all functions $f:(0,\infty)\to\mathbb{R}$ satisfying

$$\left(x+\frac{1}{x}\right)f(y)=f(xy)+f\left(\frac{y}{x}\right)$$

for all $x,y>0$.

P3 Let O be the circumcentre, and Ω be the circumcircle of an acute-angled triangle ABC. Let P be an arbitrary point on Ω, distinct from A, B, C, and their antipodes in Ω. Denote the circumcentres of the triangles AOP, BOP, and COP by O_A, O_B, and O_C, respectively. The lines ℓ_A, ℓ_B, ℓ_C perpendicular to BC, CA, and AB pass through O_A, O_B, and O_C, respectively. Prove that the circumcircle of triangle formed by ℓ_A, ℓ_B, and ℓ_C is tangent to the line OP.

– TST Practice Test 1

P1 Let $a_1,a_2,\dots,a_m$ be a set of m distinct positive even numbers and $b_1,b_2,\dots,b_n$ be a set of n distinct positive odd numbers such that

$$a_1+a_2+\cdots+a_m+b_1+b_2+\cdots+b_n=2019$$

Prove that

$$5m+12n\le 581.$$

P2 Let ABC be a triangle with $\angle A=\angle C=30^\circ$. Points D,E,F are chosen on the sides AB,BC,CA respectively so that $\angle BFD=\angle BFE=60^\circ$. Let p and p_1 be the perimeters of the triangles ABC and DEF, respectively. Prove that $p\le 2p_1$.

P3 Let $n\ge 2$ be an integer. Solve in reals:

$$|a_1-a_2|=2|a_2-a_3|=3|a_3-a_4|=\cdots=n|a_n-a_1|.$$

– TST Practice Test 2

P1 Let the points O and H be the circumcenter and orthocenter of an acute angled triangle ABC. Let D be the midpoint of BC. Let E be the point on the angle bisector of $\angle BAC$ such that $AE\perp HE$. Let F be the point such that $AEHF$ is a rectangle. Prove that D,E,F are collinear.

P2 Determine all positive integers m satisfying the condition that there exists a unique positive integer n such that there exists a rectangle which can be decomposed into n congruent squares and can also be decomposed into $m+n$ congruent squares.

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3 There are 2019 coins on a table. Some are placed with head up and others tail up. A group of 2019 persons perform the following operations: the first person chooses any one coin and then turns it over, the second person choses any two coins and turns them over and so on and the 2019-th person turns over all the coins. Prove that no matter which sides the coins are up initially, the 2019 persons can come up with a procedure for turning the coins such that all the coins have smae side up at the end of the operations.
