Moscow Mathematical Olympiad 1935

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–	tour 1
001	Find the ratio of two numbers if the ratio of their arithmetic mean to their geometric mean is $25 : 24$
002	Given the lengths of two sides of a triangle and that of the bisector of the angle between these sides, construct the triangle.
003	The base of a pyramid is an isosceles triangle with the vertex angle α. The pyramids lateral edges are at angle ϕ to the base. Find the dihedral angle θ at the edge connecting the pyramids vertex to that of angle α.
004	A train passes an observer in t_1 sec. At the same speed the train crosses a bridge ℓ m long. It takes the train t_2 sec to cross the bridge from the moment the locomotive drives onto the bridge until the last car leaves it. Find the length and speed of the train.
005	Given three parallel straight lines. Construct a square three of whose vertices belong to these lines.
006	The base of a right pyramid is a quadrilateral whose sides are each of length a. The planar angles at the vertex of the pyramid are equal to the angles between the lateral edges and the base. Find the volume of the pyramid.
007	Find four consecutive terms a, b, c, d of an arithmetic progression and four consecutive terms a_1, b_1, c_1, d_1 of a geometric progression such that $a + a_1 = 27, b + b_1 = 27, c + c_1 = 39$, and $d + d_1 = 87$.
008	Prove that if the lengths of the sides of a triangle form an arithmetic progression, then the radius of the inscribed circle is one third of one of the heights of the triangle.
009	The height of a truncated cone is equal to the radius of its base. The perimeter of a regular hexagon circumscribing its top is equal to the perimeter of an equilateral triangle inscribed in its base. Find the angle ϕ between the cones generating line and its base.
010	Solve the system $\begin{cases} x^2 + y^2 - 2z^2 = 2a^2 \\ x + y + 2z = 4(a^2 + 1) \\ z^2 - xy = a^2 \end{cases}$

011 In $\triangle ABC$, two straight lines drawn from an arbitrary point D on AB are parallel to AC and BC and intersect BC and AC at F and G, respectively. Prove that the sum of the circumferences of the circles circumscribed around $\triangle ADG$ and $\triangle BDF$ is equal to the circumference of the circle circumscribed around $\triangle ABC$.

012 The unfolding of the lateral surface of a cone is a sector of angle 120^o. The angles at the base of a pyramid constitute an arithmetic progression with a difference of 15^o. The pyramid is inscribed in the cone. Consider a lateral face of the pyramid with the smallest area. Find the angle α between the plane of this face and the base.

– tour 2

013 The median, bisector, and height, all originate at the same vertex of a triangle. Given the intersection points of the median, bisector, and height with the circumscribed circle, construct the triangle.

014 Find the locus of points on the surface of a cube that serve as the vertex of the smallest angle that subtends the diagonal.

015 Triangles $\triangle ABC$ and $\triangle A_1B_1C_1$ lie on different planes. Line AB intersects line A_1B_1, line BC intersects line B_1C_1 and line CA intersects line C_1A_1. Prove that either the three lines AA_1, BB_1, CC_1 meet at one point or that they are all parallel.

016 How many real solutions does the following system have? $\begin{cases} x+y=2 \\ xy-z^2=1 \end{cases}$

017 Solve the system $\begin{cases} x^3-y^3=26 \\ x^2y-xy^2=6 \end{cases}$

solved below

Solve the system $\begin{cases} x^3-y^3=2b \\ x^2y-xy^2=b \end{cases}$

018 Evaluate the sum: $1^3+3^3+5^3+...+(2n-1)^3$.

019 a) How many distinct ways are there are there of painting the faces of a cube six different colors?
(Colorations are considered distinct if they do not coincide when the cube is rotated.)

b)* How many distinct ways are there are there of painting the faces of a dodecahedron 12 different colors?

(Colorations are considered distinct if they do not coincide when the cube is rotated.)

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020 How many ways are there of representing a positive integer n as the sum of three positive integers?
Representations which differ only in the order of the summands are considered to be distinct.

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021 Denote by $M(a, b, c, ..., k)$ the least common multiple and by $D(a, b, c, ..., k)$ the greatest common divisor of $a, b, c, ..., k$. Prove that:
a) $M(a, b)D(a, b) = ab$,
b) $\frac{M(a,b,c)D(a,b)D(b,c)D(a,c)}{D(a,b,c)} = abc$.
