Moscow Mathematical Olympiad 1945

www.artofproblemsolving.com/community/c909511

by parmenides51

– tour 1

091 a) Divide $a^{128}-b^{128}$ by $(a+b)(a^2+b^2)(a^4+b^4)(a^8+b^8)(a^{16}+b^{16})(a^{32}+b^{32})(a^{64}+b^{64})$.
b) Divide $a^{2^k}-b^{2^k}$ by $(a+b)(a^2+b^2)(a^4+b^4)...(a^{2^{k-1}}+b^{2^{k-1}})$

092 Prove that for any positive integer $n\ge 2$ the following inequality holds:

$$\frac{1}{n+1}+\frac{1}{n+2}+...+\frac{1}{2n}>\frac{1}{2}$$

093 Find all two-digit numbers $\overline{ab}$ such that $\overline{ab}+\overline{ba}$ is a perfect square.

094 Prove that it is impossible to divide a scalene triangle into two equal triangles.

095 Two circles are tangent externally at one point. Common external tangents are drawn to them and the tangent points are connected. Prove that the sum of the lengths of the opposite sides of the quadrilateral obtained are equal.

096 Find three-digit numbers such that any its positive integer power ends with the same three digits and in the same order.

097 The system $\begin{cases} x^2-y^2=0 \\ (x-a)^2+y^2=1 \end{cases}$ generally has four solutions. For which a the number of solutions of the system is equal to three or two?

098 A right triangle ABC moves along the plane so that the vertices B and C of the triangle's acute angles slide along the sides of a given right angle. Prove that point A fills in a line segment and find its length.

– tour 2

099 Given the 6 digits: $0, 1, 2, 3, 4, 5$.
Find the sum of all even four-digit numbers which can be expressed with the help of these figures (the same figure can be repeated).

100 Suppose we have two identical cardboard polygons. We placed one polygon upon the other one and aligned. Then we pierced polygons with a pin at a point. Then we turned one of the polygons around this pin by $25^o30'$. It turned out that the polygons coincided (aligned again). What is the minimal possible number of sides of the polygons?

---

101 The side AD of a parallelogram $ABCD$ is divided into n equal segments. The nearest to A division point P is connected with B. Prove that line BP intersects the diagonal AC at point Q such that $AQ = \frac{AC}{n+1}$

---

102 Segments connect vertices A, B, C of $\triangle ABC$ with respective points A_1, B_1, C_1 on the opposite sides of the triangle. Prove that the midpoints of segments AA_1, BB_1, CC_1 do not belong to one straight line.

---

103 Solve in integers the equation $xy + 3x - 5y = -3$.

---

104 The numbers $a_1, a_2, ..., a_n$ are equal to 1 or -1. Prove that

$$2\sin\left(a_1 + \frac{a_1a_2}{2} + \frac{a_1a_2a_3}{4} + ... + \frac{a_1a_2...a_n}{2^{n-1}}\right)\frac{\pi}{4} = a_1\sqrt{2 + a_2\sqrt{2 + a_3\sqrt{2 + ... + a_n\sqrt{2}}}}$$

In particular, for $a_1 = a_2 = ... = a_n = 1$ we have

$$2\sin\left(1 + \frac{1}{2} + \frac{1}{4} + ... + \frac{1}{2^{n-1}}\right)\frac{\pi}{4} = 2\cos\frac{\pi}{2^{n+1}} = \sqrt{2 + \sqrt{2 + \sqrt{2 + ... + \sqrt{2}}}}$$

---

105 A circle rolls along a side of an equilateral triangle. The radius of the circle is equal to the height of the triangle. Prove that the measure of the arc intercepted by the sides of the triangle on this circle is equal to 60^o at all times.
