Korean MO 1994

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by dcouchman, massnet, parmenides51

– Day 1

Problem 1 Let S be the set of nonnegative integers. Find all functions $f, g, h : S \to S$ such that
$f(m+n) = g(m) + h(n)$, for all $m, n \in S$, and
$g(1) = h(1) = 1$.

Problem 2 Let α, β, γ be the angles of a triangle. Prove that $csc^2\frac{\alpha}{2} + csc^2\frac{\beta}{2} + csc^2\frac{\gamma}{2} \geq 12$
and find the conditions for equality.

Problem 3 In a triangle ABC, I and O are the incenter and circumcenter respectively, A', B', C' the excenters, and O' the circumcenter of $\triangle A'B'C'$. If R and R' are the circumradii of triangles ABC and $A'B'C'$, respectively, prove that:
(i) $R' = 2R$
(ii) $IO' = 2IO$

– Day 2

Problem 1 Consider the equation $y^2 - k = x^3$, where k is an integer.

Prove that the equation cannot have five integer solutions of the form $(x_1, y_1), (x_2, y_1 - 1), (x_3, y_1 - 2), (x_4, y_1 - 3), (x_5, y_1 - 4)$.

Also show that if it has the first four of these pairs as solutions, then $63 | k - 17$.

Problem 2 Given a set $S \subset N$ and a positive integer n, let $S \oplus \{n\} = \{s + n / s \in S\}$. The sequence S_k of sets is defined inductively as follows: $S_1 = 1$, $S_k = (S_{k-1} \oplus \{k\}) \cup \{2k - 1\}$ for $k = 2, 3, 4, ...$
(a) Determine $N - \cup_{k=1}^{\infty} S_k$.
(b) Find all n for which $1994 \in S_n$.

Problem 3 Let α, β, γ be the angles of $\triangle ABC$.
a) Show that $cos^2\alpha + cos^2\beta + cos^2\gamma = 1 - 2cos\alpha cos\beta cos\gamma$.
b) Given that $cos\alpha : cos\beta : cos\gamma = 39 : 33 : 25$, find $sin\alpha : sin\beta : sin\gamma$.