All Soviet Union Mathematical Olympiad 1990

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511 Show that $x^4 > x - \frac{1}{2}$ for all real x.

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512 The line joining the midpoints of two opposite sides of a convex quadrilateral makes equal angles with the diagonals. Show that the diagonals are equal.

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513 A graph has 30 points and each point has 6 edges. Find the total number of triples such that each pair of points is joined or each pair of points is not joined.

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514 Does there exist a rectangle which can be dissected into 15 congruent polygons which are not rectangles?
Can a square be dissected into 15 congruent polygons which are not rectangles?

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515 The point P lies inside the triangle ABC. A line is drawn through P parallel to each side of the triangle. The lines divide AB into three parts length $c, c', c"$ (in that order), and BC into three parts length $a, a', a"$ (in that order), and CA into three parts length $b, b', b"$ (in that order). Show that $abc = a'b'c' = a"b"c"$.

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516 Find three non-zero reals such that all quadratics with those numbers as coefficients have two distinct rational roots.

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517 What is the largest possible value of $|...||a_1 - a_2| - a_3| - ... - a_{1990}|$, where $a_1, a_2, ..., a_{1990}$ is a permutation of $1, 2, 3, ..., 1990$?

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518 An equilateral triangle of side n is divided into n^2 equilateral triangles of side 1. A path is drawn along the sides of the triangles which passes through each vertex just once. Prove that the path makes an acute angle at at least n vertices.

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519 Can the squares of a 1990×1990 chessboard be colored black or white so that half the squares in each row and column are black and cells symmetric with respect to the center are of opposite color?

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520 Let $x_1, x_2, ..., x_n$ be positive reals with sum 1. Show that $\frac{x_1^2}{x_1+x_2} + \frac{x_2^2}{x_2+x_3} + ... + \frac{x_{n-1}^2}{x_{n-1}+x_n} + \frac{x_n^2}{x_n+x_1} \geq \frac{1}{2}$.

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521 $ABCD$ is a convex quadrilateral. X is a point on the side $AB.AC$ and DX intersect at Y. Show that the circumcircles of ABC, CDY and BDX have a common point.

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522 Two grasshoppers sit at opposite ends of the interval $[0, 1]$. A finite number of points (greater than zero) in the interval are marked. A move is for a grasshopper to select a marked point and jump over it to the equidistant point the other side. This point must lie in the interval for the move to be allowed, but it does not have to be marked. What is the smallest n such that if each grasshopper makes n moves or less, then they end up with no marked points between them?

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523 Find all integers n such that $\left[\frac{n}{1!}\right] + \left[\frac{n}{2!}\right] + ... + \left[\frac{n}{10!}\right] = 1001.$

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524 A, B, C are adjacent vertices of a regular $2n$-gon and D is the vertex opposite to B (so that BD passes through the center of the $2n$-gon). X is a point on the side AB and Y is a point on the side BC so that angle $XDY = \frac{\pi}{2n}$. Show that DY bisects angle $\angle XYC$.

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525 A graph has n points and $\frac{n(n-1)}{2}$ edges. Each edge is colored with one of k colors so that there are no closed monochrome paths. What is the largest possible value of n (given k)?

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526 Given a point X and n vectors $\overrightarrow{x_i}$ with sum zero in the plane. For each permutation of the vectors we form a set of n points, by starting at X and adding the vectors in order. For example, with the original ordering we get X_1 such that $XX_1 = \overrightarrow{x_1}, X_2$ such that $X_1X_2 = \overrightarrow{x_2}$ and so on. Show that for some permutation we can find two points Y, Z with angle $\angle YXZ = 60^o$, so that all the points lie inside or on the triangle XYZ.

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527 Two unequal circles intersect at X and Y. Their common tangents intersect at Z. One of the tangents touches the circles at P and Q. Show that ZX is tangent to the circumcircle of PXQ.

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528 Given 1990 piles of stones, containing $1, 2, 3, ..., 1990$ stones. A move is to take an equal number of stones from one or more piles. How many moves are needed to take all the stones?

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529 A quadratic polynomial $p(x)$ has positive real coefficients with sum 1. Show that given any positive real numbers with product 1, the product of their values under p is at least 1.

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530 A cube side 100 is divided into a million unit cubes with faces parallel to the large cube. The edges form a lattice. A prong is any three unit edges with a common vertex. Can we decompose the lattice into prongs with no common edges?

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531 For which positive integers n is $3^{2n+1} - 2^{2n+1} - 6^n$ composite?

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532 If every altitude of a tetrahedron is at least 1, show that the shortest distance between each pair of opposite edges is more than 2.

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533 A game is played in three moves. The first player picks any real number, then the second player makes it the coefficient of a cubic, except that the coefficient of x^3 is already fixed at 1. Can the first player make his choices so that the final cubic has three distinct integer roots?

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534 Given $2n$ genuine coins and $2n$ fake coins. The fake coins look the same as genuine coins but weigh less (but all fake coins have the same weight). Show how to identify each coin as genuine or fake using a balance at most $3n$ times.
