

All Soviet Union Mathematical Olympiad 1991

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by parmenides51

535 Find all integers a, b, c, d such that

$$\begin{cases} ab - 2cd = 3 \\ ac + bd = 1 \end{cases}$$

536 n numbers are written on a blackboard. Someone then repeatedly erases two numbers and writes half their arithmetic mean instead, until only a single number remains. If all the original numbers were 1, show that the final number is not less than $\frac{1}{n}$.

537 Four lines in the plane intersect in six points. Each line is thus divided into two segments and two rays. Is it possible for the eight segments to have lengths $1, 2, 3, \dots, 8$? Can the lengths of the eight segments be eight distinct integers?

538 A lottery ticket has 50 cells into which one must put a permutation of $1, 2, 3, \dots, 50$. Any ticket with at least one cell matching the winning permutation wins a prize. How many tickets are needed to be sure of winning a prize?

539 Find unequal integers m, n such that $mn + n$ and $mn + m$ are both squares. Can you find such integers between 988 and 1991?

540 $ABCD$ is a rectangle. Points K, L, M, N are chosen on AB, BC, CD, DA respectively so that KL is parallel to MN , and KM is perpendicular to LN . Show that the intersection of KM and LN lies on BD .

541 An investigator works out that he needs to ask at most 91 questions on the basis that all the answers will be yes or no and all will be true. The questions may depend upon the earlier answers. Show that he can make do with 105 questions if at most one answer could be a lie.

542 A minus sign is placed on one square of a 5×5 board and plus signs are placed on the remaining squares. A move is to select a $2 \times 2, 3 \times 3, 4 \times 4$ or 5×5 square and change all the signs in it. Which initial positions allow a series of moves to change all the signs to plus?

543 Show that $\frac{(x+y+z)^2}{3} \geq x\sqrt{yz} + y\sqrt{zx} + z\sqrt{xy}$ for all non-negative reals x, y, z .

544 Does there exist a triangle in which two sides are integer multiples of the median to that side? Does there exist a triangle in which every side is an integer multiple of the median to that side?

- 545** The numbers $1, 2, 3, \dots, n$ are written on a blackboard (where $n \geq 3$). A move is to replace two numbers by their sum and non-negative difference. A series of moves makes all the numbers equal k . Find all possible k .
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- 546** The figure below is cut along the lines into polygons (which need not be convex). No polygon contains a 2×2 square. What is the smallest possible number of polygons?
[missing figure]
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- 547** ABC is an acute-angled triangle with circumcenter O . The circumcircle of ABO intersects AC and BC at M and N . Show that the circumradii of ABO and MNC are the same.
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- 548** A polygon can be transformed into a new polygon by making a straight cut, which creates two new pieces each with a new edge. One piece is then turned over and the two new edges are reattached. Can repeated transformations of this type turn a square into a triangle?
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- 549** An $h \times k$ minor of an $n \times n$ table is the hk cells which lie in h rows and k columns. The semiperimeter of the minor is $h+k$. A number of minors each with semiperimeter at least n together include all the cells on the main diagonal. Show that they include at least half the cells in the table.
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- 550** a) $r_1, r_2, \dots, r_{100}, c_1, c_2, \dots, c_{100}$ are distinct reals. The number $r_i + c_j$ is written in position i, j of a 100×100 array. The product of the numbers in each column is 1. Show that the product of the numbers in each row is -1 .
b) $r_1, r_2, \dots, r_{2n}, c_1, c_2, \dots, c_{2n}$ are distinct reals. The number $r_i + c_j$ is written in position i, j of a $2n \times 2n$ array. The product of the numbers in each column is the same. Show that the product of the numbers in each row is also the same.
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- 551** A sequence of positive integers is constructed as follows. If the last digit of a_n is greater than 5, then a_{n+1} is $9a_n$. If the last digit of a_n is 5 or less and a_n has more than one digit, then a_{n+1} is obtained from a_n by deleting the last digit. If a_n has only one digit, which is 5 or less, then the sequence terminates. Can we choose the first member of the sequence so that it does not terminate?
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- 552** $p(x)$ is the cubic $x^3 - 3x^2 + 5x$. If h is a real root of $p(x) = 1$ and k is a real root of $p(x) = 5$, find $h + k$.
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- 553** The chords AB and CD of a sphere intersect at X . A, C and X are equidistant from a point Y on the sphere. Show that BD and XY are perpendicular.
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- 554** Do there exist 4 vectors in the plane so that none is a multiple of another, but the sum of each pair is perpendicular to the sum of the other two? Do there exist 91 non-zero vectors in the plane such that the sum of any 19 is perpendicular to the sum of the others?
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- 555** $ABCD$ is a square. The points X on the side AB and Y on the side AD are such that $AX \cdot AY = 2BX \cdot DY$. The lines CX and CY meet the diagonal BD in two points. Show that these points lie on the circumcircle of AXY .
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- 556** X is a set with 100 members. What is the smallest number of subsets of X such that every pair of elements belongs to at least one subset and no subset has more than 50 members? What is the smallest number if we also require that the union of any two subsets has at most 80 members?
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- 557** The real numbers $x_1, x_2, \dots, x_{1991}$ satisfy $|x_1 - x_2| + |x_2 - x_3| + \dots + |x_{1990} - x_{1991}| = 1991$. What is the maximum possible value of $|s_1 - s_2| + |s_2 - s_3| + \dots + |s_{1990} - s_{1991}|$, where $s_n = \frac{x_1 + x_2 + \dots + x_n}{n}$?
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