Commonwealth of Independent States 1992 (ASU)
www.artofproblemsolving.com/community/c909856
by parmenides51

558 Show that $x^4 + y^4 + z^2 \geq xyz\sqrt{8}$ for all positive reals x, y, z.

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559 E is a point on the diagonal BD of the square $ABCD$. Show that the points A, E and the circumcenters of ABE and ADE form a square.

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560 A country contains n cities and some towns. There is at most one road between each pair of towns and at most one road between each town and each city, but all the towns and cities are connected, directly or indirectly. We call a route between a city and a town a gold route if there is no other route between them which passes through fewer towns. Show that we can divide the towns and cities between n republics, so that each belongs to just one republic, each republic has just one city, and each republic contains all the towns on at least one of the gold routes between each of its towns and its city.

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561 Given an infinite sheet of square ruled paper. Some of the squares contain a piece. A move consists of a piece jumping over a piece on a neighbouring square (which shares a side) onto an empty square and removing the piece jumped over. Initially, there are no pieces except in an mxn rectangle ($m, n > 1$) which has a piece on each square. What is the smallest number of pieces that can be left after a series of moves?

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562 Does there exist a 4-digit integer which cannot be changed into a multiple of 1992 by changing 3 of its digits?

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563 A and B lie on a circle. P lies on the minor arc AB. Q and R (distinct from P) also lie on the circle, so that P and Q are equidistant from A, and P and R are equidistant from B. Show that the intersection of AR and BQ is the reflection of P in AB.

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564 Find all real x, y such that $\begin{cases} (1+x)(1+x^2)(1+x^4) = 1+y^7 \\ (1+y)(1+y^2)(1+y^4) = 1+x^7 \end{cases}$.

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565 An $m \times n$ rectangle is divided into mn unit squares by lines parallel to its sides. A gnomon is the figure of three unit squares formed by deleting one unit square from a 2×2 square. For what m, n can we divide the rectangle into gnomons so that no two gnomons form a rectangle and no vertex is in four gnomons?

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566 Show that for any real numbers $x, y > 1$, we have $\frac{x^2}{y-1} + \frac{y^2}{x-1} \geq 8$

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567 Show that if 15 numbers lie between 2 and 1992 and each pair is coprime, then at least one is prime.

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568 A cinema has its seats arranged in n rows $\times m$ columns. It sold mn tickets but sold some seats more than once. The usher managed to allocate seats so that every ticket holder was in the correct row or column. Show that he could have allocated seats so that every ticket holder was in the correct row or column and at least one person was in the correct seat. What is the maximum k such that he could have always put every ticket holder in the correct row or column and at least k people in the correct seat?

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569 Circles C and C' intersect at O and X. A circle center O meets C at Q and R and meets C' at P and S. PR and QS meet at Y distinct from X. Show that $\angle YXO = 90^o$.

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570 Define the sequence $a_1 = 1, a_2, a_3, ...$ by $a_{n+1} = a_1^2 + a_2^2 + a_3^2 + ... + a_n^2 + n.$
Show that 1 is the only square in the sequence.

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571 $ABCD$ is a parallelogram. The excircle of ABC opposite A has center E and touches the line AB at X. The excircle of ADC opposite A has center F and touches the line AD at Y. The line FC meets the lineAB at W, and the line EC meets the line AD at Z. Show that $WX = YZ$.

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572 Half the cells of a $2m \times n$ board are colored black and the other half are colored white. The cells at the opposite ends of the main diagonal are different colors. The center of each black cell is connected to the center of every other black cell by a straight line segment, and similarly for the white cells. Show that we can place an arrow on each segment so that it becomes a vector and the vectors sum to zero.

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573 A graph has 17 points and each point has 4 edges.
Show that there are two points which are not joined and which are not both joined to the same point.

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574 Let $f(x) = a\cos(x + 1) + b\cos(x + 2) + c\cos(x + 3)$, where a, b, c are real.
Given that $f(x)$ has at least two zeros in the interval $(0, \pi)$, find all its real zeros.

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575 A plane intersects a sphere in a circle C. The points A and B lie on the sphere on opposite sides of the plane. The line joining A to the center of the sphere is normal to the plane. Another plane p intersects the segment AB and meets C at P and Q. Show that $BP \cdot BQ$ is independent of the choice of p.

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576 If you have an algorithm for finding all the real zeros of any cubic polynomial, how do you find the real solutions to $x = p(y), y = p(x)$, where p is a cubic polynomial?

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577 Find all integers $k > 1$ such that for some distinct positive integers a, b, the number $k^a + 1$ can be obtained from $k^b + 1$ by reversing the order of its (decimal) digits.

578 An equilateral triangle side 10 is divided into 100 equilateral triangles of side 1 by lines parallel to its sides. There are m equilateral tiles of 4 unit triangles and $25 - m$ straight tiles of 4 unit triangles (as shown below). For which values of m can they be used to tile the original triangle. [The straight tiles may be turned over.]

579 1992 vectors are given in the plane. Two players pick unpicked vectors alternately. The winner is the one whose vectors sum to a vector with larger magnitude (or they draw if the magnitudes are the same). Can the first player always avoid losing?

580 If $a > b > c > d > 0$ are integers such that $ad = bc$, show that $(a - d)^2 \geq 4d + 8$.