Mediterranean Mathematics Olympiad 2019

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by parmenides51

1 Let ΔABC be a triangle with angle $\angle CAB=60^\circ$, let D be the intersection point of the angle bisector at A and the side BC, and let r_B,r_C,r be the respective radii of the incircles of ABD, ADC, ABC. Let b and c be the lengths of sides AC and AB of the triangle. Prove that

$$\frac{1}{r_B}+\frac{1}{r_C} ~=~ 2\cdot\left(\frac{1}{r}+\frac{1}{b}+\frac{1}{c} \right)$$

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2 Let $m_1<m_2<\cdots<m_s$ be a sequence of $s\ge2$ positive integers, none of which can be written as the sum of (two or more) distinct other numbers in the sequence. For every integer r with $1\le r<s$, prove that

$$r\cdot m_r+m_s ~\ge~ (r+1)(s-1).$$

(Proposed by Gerhard Woeginger, Austria)

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3 Prove that there exist infinitely many positive integers x,y,z for which the sum of the digits in the decimal representation of $~4x^4+y^4-z^2+4xyz~$ is at most 2.

(Proposed by Gerhard Woeginger, Austria)

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4 Let P be a point in the interior of an equilateral triangle with height 1, and let x,y,z denote the distances from P to the three sides of the triangle. Prove that

$$x^2+y^2+z^2 ~\ge~ x^3+y^3+z^3+6xyz$$
