Danube Mathematical Competition 2013

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by parmenides51

– Junior

1 Determine the natural numbers $n \geq 2$ for which exist $x_1, x_2, ..., x_n \in R^*$, such that

$$x_1 + x_2 + ... + x_n = \frac{1}{x_1} + \frac{1}{x_2} + ... + \frac{1}{x_n} = 0$$

2 Consider 64 distinct natural numbers, at most equal to 2012. Show that it is possible to choose four of them, denoted as a, b, c, d such that $a + b - c - d$ to be a multiple of 2013

3 Determine the natural numbers m, n such as $85^m - n^4 = 4$

4 Let $ABCD$ be a rectangle with $AB \neq BC$ and the center the point O. Perpendicular from O on BD intersects lines AB and BC in points E and F respectively. Points M and N are midpoints of segments $[CD]$ and $[AD]$ respectively. Prove that $FM \perp EN$.

– Senior

1 Given six points on a circle, A, a, B, b, C, c, show that the Pascal lines of the hexagrams $AaBbCc$, $AbBcCa$, Ac are concurrent.

2 Let a, b, c, n be four integers, where n≥ 2, and let p be a prime dividing both $a^2 + ab + b^2$ and $a^n + b^n + c^n$, but not $a + b + c$. for instance, $a \equiv b \equiv -1 (mod\ 3), c \equiv 1 (mod\ 3), n$ a positive even integer, and $p = 3$ or $a = 4, b = 7, c = -13, n = 5$, and $p = 31$ satisfy these conditions. Show that n and $p - 1$ are not coprime.

3 Show that, for every integer $r \geq 2$, there exists an r-chromatic simple graph (no loops, nor multiple edges) which has no cycle of less than 6 edges

4 Show that there exists a proper non-empty subset S of the set of real numbers such that, for every real number x, the set $\{nx + S : n \in N\}$ is finite, where $nx + S = \{nx + s : s \in S\}$