Danube Mathematical Competition 2014

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by parmenides51

–	Junior	
1	Determine the natural number $a = \frac{p+q}{r} + \frac{q+r}{p} + \frac{r+p}{q}$ where p, q and r are prime positive numbers.	
2	We call *word* a sequence of letters $\overline{l_1 l_2 ... l_n}, n \geq 1$. A *word* $\overline{l_1 l_2 ... l_n}, n \geq 1$ is called *palindrome* if $l_k = l_{n-k+1}$, for any $k, 1 \leq k \leq n$. Consider a *word* $X = \overline{l_1 l_2 ... l_{2014}}$ in which $l_k \in \{A, B\}$, for any $k, 1 \leq k \leq 2014$. Prove that there are at least 806 *palindrome words* to "stick" together to get word X.	
3	Let ABC be a triangle with $\angle A < 90^o, AB \neq AC$. Denote H the orthocenter of triangle ABC, N the midpoint of segment $[AH]$, M the midpoint of segment $[BC]$ and D the intersection point of the angle bisector of $\angle BAC$ with the segment $[MN]$. Prove that $< ADH = 90^o$	
4	Consider the real numbers $a_1, a_2, ..., a_{2n}$ whose sum is equal to 0. Prove that among pairs $(a_i, a_j), i < j$ where $i, j \in \{1, 2, ..., 2n\}$ .there are at least $2n - 1$ pairs with the property that $a_i + a_j \geq 0$.	
–	Senior	
1	Two circles γ_1 and γ_2 cross one another at two points; let A be one of these points. The tangent to γ_1 at A meets again γ_2 at B, the tangent to γ_2 at A meets again γ_1 at C, and the line BC meets again γ_1 and γ_2 at D_1 and D_2, respectively. Let E_1 and E_2 be interior points of the segments AD_1 and AD_2, respectively, such that $AE_1 = AE_2$. The lines BE_1 and AC meet at M, the lines CE_2 and AB meet at N, and the lines MN and BC meet at P. Show that the line PA is tangent to the circle ABC.	
2	Let S be a set of positive integers such that $\lfloor \sqrt{x} \rfloor = \lfloor \sqrt{y} \rfloor$ for all $x, y \in S$. Show that the products xy, where $x, y \in S$, are pairwise distinct.	
3	Given any integer $n \geq 2$, show that there exists a set of n pairwise coprime composite integers in arithmetic progression.	
4	Let n be a positive integer and let $\triangle$ be the closed triangular domain with vertices at the lattice points $(0, 0), (n, 0)$ and $(0, n)$. Determine the maximal cardinality a set S of lattice points in $\triangle$ may have, if the line through every pair of distinct points in S is parallel to no side of $\triangle$.	