Danube Mathematical Competition 2015

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–	Junior	
1	Consider a positive integer $n=\overline{a_1a_2...a_k}, k \geq 2$.A *trunk* of n is a number of the form $\overline{a_1a_2...a_t}, 1 \leq t \leq k-1$.(For example,the number 23 is a *trunk* of 2351.) By $T(n)$ we denote the sum of all *trunk* of n and let $S(n)=a_1+a_2+...+a_k$.Prove that $n=S(n)+9\cdot T(n)$.	
2	Consider the set $A=\{1,2,...,120\}$ and M a subset of A such that $\vert M\vert =30$.Prove that there are 5 different subsets of M,each of them having two elements,such that the absolute value of the difference of the elements of each subset is the same.	
3	Solve in N $a^2=2^b3^c+1$.	
4	Let $ABCD$ be a rectangle with $AB \ge BC$ Point M is located on the side (AD), and the perpendicular bisector of $[MC]$ intersects the line BC at the point N. Let $Q = MN \cup AB$. Knowing that $\angle MQA = 2 \cdot \angle BCQ$, show that the quadrilateral $ABCD$ is a square.	
5	A lantern needs exactly 2 charged batteries in order to work.We have available n charged batteries and n uncharged batteries,$n \geq 4$(all batteries look the same). A *try* consists in introducing two batteries in the lantern and verifying if the lantern works.Prove that we can find a pair of charged batteries in at most $n+2$ *tries*.	
–	Senior	
1	Let $ABCD$ be a cyclic quadrangle, let the diagonals AC and BD cross at O, and let I and J be the incentres of the triangles ABC and ABD, respectively. The line IJ crosses the segments OA and OB at M and N, respectively. Prove that the triangle OMN is isosceles.	
2	Show that the edges of a connected simple (no loops and no multiple edges) finite graph can be oriented so that the number of edges leaving each vertex is even if and only if the total number of edges is even	
3	Determine all positive integers n such that all positive integers less than or equal to n and relatively prime to n are pairwise coprime.	
4	Given an integer $n\ge 2$,determine the numbers that written in the form $a_1a_2+a_2a_3+...a_{k-1}a_k$, where k is an integer greater than or equal to 2, and a_1 ,... a_k are positive integers with sum n.	