Danube Mathematical Competition 2018

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–	Junior
1	Find all the pairs (n, m) of positive integers which fulfil simultaneously the conditions: i) the number n is composite; ii) if the numbers $d_1, d_2, ..., d_k, k \in N^*$ are all the proper divisors of n, then the numbers $d_1 + 1, d_2 + 1, ..., d_k + 1$ are all the proper divisors of m.
2	Let ABC be a triangle such that in its interior there exists a point D with $\angle DAC = \angle DCA = 30^o$ and $\angle DBA = 60^o$. Denote E the midpoint of the segment BC, and take F on the segment AC so that $AF = 2FC$. Prove that $DE \perp EF$.
3	Find all the positive integers n with the property: there exists an integer $k > 2$ and the positive rational numbers $a_1, a_2, ..., a_k$ such that $a_1 + a_2 + .. + a_k = a_1 a_2 ... a_k = n$.
4	Let M be the set of positive odd integers. For every positive integer n, denote $A(n)$ the number of the subsets of M whose sum of elements equals n. For instance, $A(9) = 2$, because there are exactly two subsets of M with the sum of their elements equal to 9: $\{9\}$ and $\{1, 3, 5\}$. a) Prove that $A(n) \leq A(n + 1)$ for every integer $n \geq 2$. b) Find all the integers $n \geq 2$ such that $A(n) = A(n + 1)$
–	Senior
1	Suppose we have a necklace of n beads. Each bead is labeled with an integer and the sum of all these labels is $n - 1$. Prove that we can cut the necklace to form a string, whose consecutive labels $x_1, x_2, ..., x_n$ satisfy $\sum_{i=1}^{k} x_i \leq k - 1$ for any $k = 1, ..., n$
2	Prove that there are in finitely many pairs of positive integers (m, n) such that simultaneously m divides $n^2 + 1$ and n divides $m^2 + 1$.
3	Let ABC be an acute non isosceles triangle. The angle bisector of angle A meets again the circumcircle of the triangle ABC in D. Let O be the circumcenter of the triangle ABC. The angle bisectors of $\angle AOB$, and $\angle AOC$ meet the circle γ of diameter AD in P and Q respectively. The line PQ meets the perpendicular bisector of AD in R. Prove that $AR // BC$.

4 Let $n \geq 3$ be an odd number and suppose that each square in a $n \times n$ chessboard is colored either black or white. Two squares are considered adjacent if they are of the same color and share a common vertex and two squares a, b are considered connected if there exists a sequence of squares $c_1, \dots, c_k$ with $c_1 = a, c_k = b$ such that c_i, c_{i+1} are adjacent for $i = 1, 2, \dots, k-1$.

Find the maximal number M such that there exists a coloring admitting M pairwise disconnected squares.