Final Round - 2008

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– Day 1

1 Hexagon $ABCDEF$ is inscribed in a circle O.
Let $BD \cap CF = G, AC \cap BE = H, AD \cap CE = I$
Following conditions are satisfied. $BD \perp CF, CI = AI$

Prove that $CH = AH + DE$ is equivalent to $GH \times BD = BC \times DE$

2 Find all integer polynomials f such that there are infinitely many pairs of relatively prime natural numbers (a, b) so that $a + b \mid f(a) + f(b)$.

3 Determine all functions $f : \mathbb{R}^+ \to \mathbb{R}$ that satisfy the following $f(1) = 2008$, $|f(x)| \le x^2 + 1004^2$, $f\left(x + y + \frac{1}{x} + \frac{1}{y}\right) = f\left(x + \frac{1}{y}\right) + f\left(y + \frac{1}{x}\right)$.

– Day 2

4 For any positive integer $m \ge 2$ define $A_m = \{m+1, 3m+2, 5m+3, 7m+4, \dots, (2k-1)m+k, \dots\}$.

(1) For every $m \ge 2$, prove that there exists a positive integer a that satisfies $1 \le a < m$ and $2^a \in A_m$ or $2^a + 1 \in A_m$.

(2) For a certain $m \ge 2$, let a, b be positive integers that satisfy $2^a \in A_m$, $2^b + 1 \in A_m$. Let a_0, b_0 be the least such pair a, b.
Find the relation between a_0 and b_0.

5 Quadrilateral $ABCD$ is inscribed in a circle O.
Let $AB \cap CD = E$ and $P \in BC, EP \perp BC$, $R \in AD, ER \perp AD$, $EP \cap AD = Q, ER \cap BC = S$
Let K be the midpoint of QS

Prove that E, K, O are collinear.

6 There is $n \times n$ chessboard. Each square has a number between 0 and k. There is a button for each row and column, which increases the number of n numbers of the row or column the button represents(if the number of the square is k, then it becomes 0). If certain button is pressed, call it 'operation.'

And we have a chessboard which is filled with 0(for all squares). After some 'operation's, the numbers of squares are different now. Prove that we can make all of the number 0 within kn 'operation's.