Tuymaada Olympiad 2019

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–	Juniors
–	Day 1
1	In a sequence $a_1, a_2, ..$ of real numbers the product $a_1 a_2$ is negative, and to define a_n for $n > 2$ one pair (i, j) is chosen among all the pairs $(i, j), 1 \leq i < j < n$, not chosen before, so that $a_i + a_j$ has minimum absolute value, and then a_n is set equal to $a_i + a_j$. Prove that $\lvert a_i \rvert < 1$ for some i.
2	A triangle ABC with $AB < AC$ is inscribed in a circle ω. Circles γ_1 and γ_2 touch the lines AB and AC, and their centres lie on the circumference of ω. Prove that C lies on a common external tangent to γ_1 and γ_2.
3	The plan of a picture gallery is a chequered figure where each square is a room, and every room can be reached from each other by moving to rooms adjacent by side. A custodian in a room can watch all the rooms that can be reached from this room by one move of a chess rook (without leaving the gallery). What minimum number of custodians is sufficient to watch all the rooms in every gallery of n rooms ($n > 1$)?
4	A quota of diplomas at the All-Russian Olympiad should be strictly less than 45%. More than 20 students took part in the olympiad. After the olympiad the Authorities declared the results low because the quota of diplomas was significantly less than 45%. The Jury responded that the quota was already maximum possible on this olympiad or any other olympiad with smaller number of participants. Then the Authorities ordered to increase the number of participants for the next olympiad so that the quota of diplomas became at least two times closer to 45%. Prove that the number of participants should be at least doubled.
–	Day 2
5	Is it possible to draw in the plane the graph presented in the figure so that all the vertices are different points and all the edges are unit segments? (The segments can intersect at points different from vertices.)
6	Let $\mathbb{S}$ is the set of prime numbers that less or equal to 26. Is there any $a_1, a_2, a_3, a_4, a_5, a_6 \in \mathbb{N}$ such that $$gcd(a_i, a_j) \in \mathbb{S} \qquad \text{for } 1 \leq i \neq j \leq 6$$

and for every element p of $\mathbb{S}$ there exists a pair of $1 \leq k \neq l \leq 6$ such that

$$s = gcd(a_k, a_l)?$$

7 A circle ω touches the sides AB and BC of a triangle ABC and intersects its side AC at K. It is known that the tangent to ω at K is symmetrical to the line AC with respect to the line BK. What can be the difference $AK - CK$ if $AB = 9$ and $BC = 11$?

8 Andy, Bess, Charley and Dick play on a 1000×1000 board. They make moves in turn: Andy first, then Bess, then Charley and finally Dick, after that Andy moves again and so on. At each move a player must paint several unpainted squares forming $2 \times 1, 1 \times 2, 1 \times 3$, or 3×1 rectangle. The player that cannot move loses. Prove that some three players can cooperate to make the fourth player lose.

– Seniors

– Day 1

1 same as juniors Q1

2 A trapezoid $ABCD$ with $BC//AD$ is given. The points B' and C' are symmetrical to B and C with respect to CD and AB, respectively. Prove that the midpoint of the segment joining the circumcentres of ABC' and $B'CD$ is equidistant from A and D.

3 The plan of a picture gallery is a chequered figure where each square is a room, and every room can be reached from each other by moving to adjacent rooms. A custodian in a room can watch all the rooms that can be reached from this room by one move of a chess queen (without leaving the gallery). What minimum number of custodians is sufficient to watch all the rooms in every gallery of n rooms ($n > 2$)?

4 A calculator can square a number or add 1 to it. It cannot add 1 two times in a row. By several operations it transformed a number x into a number $S > x^n + 1$ (x, n, S are positive integers). Prove that $S > x^n + x - 1$.

– Day 2

5 same as juniors Q6

6 Prove that the expression

$$(1^4 + 1^2 + 1)(2^4 + 2^2 + 1) \dots (n^4 + n^2 + 1)$$

is not square for all $n \in \mathbb{N}$

7 N cells chosen on a rectangular grid. Let a_i is number of chosen cells in i-th row, b_j is number of chosen cells in j-th column. Prove that

$$\prod_i a_i! \cdot \prod_j b_j! \leq N!$$

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8 In $\triangle ABC$ $\angle B$ is obtuse and $AB \neq BC$. Let O is the circumcenter and ω is the circumcircle of this triangle. N is the midpoint of arc ABC. The circumcircle of $\triangle BON$ intersects AC on points X and Y. Let $BX \cap \omega = P \neq B$ and $BY \cap \omega = Q \neq B$. Prove that P, Q and reflection of N with respect to line AC are collinear.
