South Africa National Olympiad 2019

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by DylanN

1	Determine all positive integers a for which a^a is divisible by 20^{19}.

2	We have a deck of 90 cards that are numbered from 10 to 99 (all two-digit numbers). How many sets of three or more different cards in this deck are there such that the number on one of them is the sum of the other numbers, and those other numbers are consecutive?

3	Let A, B, C be points on a circle whose centre is O and whose radius is 1, such that $\angle BAC = 45°$. Lines AC and BO (possibly extended) intersect at D, and lines AB and CO (possibly extended) intersect at E. Prove that $BD \cdot CE = 2$.

4	The squares of an 8×8 board are coloured alternatingly black and white. A rectangle consisting of some of the squares of the board is called *important* if its sides are parallel to the sides of the board and all its corner squares are coloured black. The side lengths can be anything from 1 to 8 squares. On each of the 64 squares of the board, we write the number of important rectangles in which it is contained. The sum of the numbers on the black squares is B, and the sum of the numbers on the white squares is W. Determine the difference $B - W$.

5	Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that

$$f(a^3) + f(b^3) + f(c^3) + 3f(a+b)f(b+c)f(c+a) = (f(a+b+c))^3$$

for all integers a, b, c.

6	Determine all pairs (m, n) of non-negative integers that satisfy the equation

$$20^m - 10m^2 + 1 = 19^n.$$