Kosovo Team Selection Test 2019

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1 There are 2019 cards in a box. Each card has a number written on one of its sides and a letter on the other side. Amy and Ben play the following game: in the beginning Amy takes all the cards, places them on a line and then she flips as many cards as she wishes. Each time Ben touches a card he has to flip it and its neighboring cards. Ben is allowed to have as many as 2019 touches. Ben wins if all the cards are on the numbers' side, otherwise Amy wins. Determine who has a winning strategy.

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2 Determine all functions $f:\mathbb{R}\rightarrow\mathbb{R}$ such that for every $x,y\in\mathbb{R}$

$$f(x^{4}-y^{4})+4f(xy)^{2}=f(x^{4}+y^{4})$$

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3 Prove that there exist infinitely many positive integers n such that $\frac{4^n+2^n+1}{n^2+n+1}$ is a positive integer.

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4 Given a rectangle $ABCD$ such that $AB=b>2a=BC$, let E be the midpoint of AD. On a line parallel to AB through point E, a point G is chosen such that the area of GCE is

$$(GCE)=\frac{1}{2}\left(\frac{a^3}{b}+ab\right)$$

Point H is the foot of the perpendicular from E to GD and a point I is taken on the diagonal AC such that the triangles ACE and AEI are similar. The lines BH and IE intersect at K and the lines CA and EH intersect at J. Prove that $KJ\perp AB$.

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5 a,b,c,d are fixed positive real numbers. Find the maximum value of the function $f:\mathbb{R^{+}}_{0}\rightarrow\mathbb{R}$ $f(x)=\frac{a+bx}{b+cx}+\frac{b+cx}{c+dx}+\frac{c+dx}{d+ax}+\frac{d+ax}{a+bx},x\geq 0$
