Balkan MO Shortlist 2018

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by parmenides51, sqing, Hamel, dangerousliri, XbenX, ThE-dArK-lOrD, microsoft_office_word, Wictro

–   Algebra

A1   Let a, b, c be positive real numbers such that $abc = \frac{2}{3}$. Prove that:

$$\frac{ab}{a+b} + \frac{bc}{b+c} + \frac{ca}{c+a} \geqslant \frac{a+b+c}{a^3+b^3+c^3}.$$

A2   Let q be a positive rational number. Two ants are initially at the same point X in the plane. In the n-th minute $(n = 1, 2, ...)$ each of them chooses whether to walk due north, east, south or west and then walks the distance of q^n metres. After a whole number of minutes, they are at the same point in the plane (not necessarily X), but have not taken exactly the same route within that time. Determine all possible values of q.

Proposed by Jeremy King, UK

A3   Show that for every positive integer n we have:

$$\sum_{k=0}^{n}\left(\frac{2n+1-k}{k+1}\right)^k = \left(\frac{2n+1}{1}\right)^0 + \left(\frac{2n}{2}\right)^1 + ... + \left(\frac{n+1}{n+1}\right)^n \leq 2^n$$

Proposed by Dorlir Ahmeti, Albania

A4   Let a, b, c be positive real numbers such that $abc = 1$. Prove that:

$$2(a^2 + b^2 + c^2)\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right) \geq 3(a + b + c + ab + bc + ca).$$

A5   Let $f : \mathbb{R} \to \mathbb{R}$ be a concave function and $g : \mathbb{R} \to \mathbb{R}$ be a continuous function . If

$$f(x+y) + f(x-y) - 2f(x) = g(x)y^2$$

for all $x, y \in \mathbb{R}$, prove that f is a second degree polynomial.

A6   Let $x_1, x_2, \cdots, x_n$ be positive real numbers . Prove that:

$$\sum_{i=1}^{n} x_i^2 \geq \frac{1}{n+1}\left(\sum_{i=1}^{n} x_i\right)^2 + \frac{12(\sum_{i=1}^{n} ix_i)^2}{n(n+1)(n+2)(3n+1)}.$$

–		Combinatorics

C1	Let N be an odd number, $N \geq 3$. N tennis players take part in a championship. Before starting the championship, a commission puts the players in a row depending on how good they think the players are. During the championship, every player plays with every other player exactly once, and each match has a winner. A match is called *suprising* if the winner was rated lower by the commission. At the end of the tournament, players are arranged in a line based on the number of victories they have achieved. In the event of a tie, the commission's initial order is used to decide which player will be higher.

It turns out that the final order is exactly the same as the commission's initial order. What is the maximal number of suprising matches that could have happened.

C2	Alice and Bob play the following game: They start with non-empty piles of coins. Taking turns, with Alice playing first, each player choose a pile with an even number of coins and moves half of the coins of this pile to the other pile. The game ends if a player cannot move, in which case the other player wins.

Determine all pairs (a, b) of positive integers such that if initially the two piles have a and b coins respectively, then Bob has a winning strategy.

Proposed by Dimitris Christophides, Cyprus

C3	An open necklace can contain rubies, emeralds, and sapphires. At every step we can perform any of the following operations:

-We can replace two consecutive rubies with an emerald and a sapphire, where the emerald is on the left of the sapphire.
-We can replace three consecutive emeralds with a sapphire and a ruby, where the sapphire is on the left of the ruby.
-If we find two consecutive sapphires then we can remove them.
-If we find consecutively and in this order a ruby, an emerald, and a sapphire, then we can remove them.

Furthermore we can also reverse all of the above operations. For example by reversing 3. we can put two consecutive sapphires on any position we wish.

Initially the necklace has one sapphire (and no other precious stones). Decide, with proof, whether there is a finite sequence of steps such that at the end of this sequence the necklace contains one emerald (and no other precious stones).

Remark: A necklace is open if its precious stones are on a line from left to right. We are not allowed to move a precious stone from the rightmost position to the leftmost as we would be able to do if the necklace was closed.

Proposed by Demetres Christofides, Cyprus

–   Geometry

G1   Let ABC be an acute triangle and let M be the midpoint of side BC. Let D, E be the excircles of triangles AMB, AMC respectively, towards M. Circumcirscribed circle of triangle ABD intersects line BC at points B and F. Circumcirscribed circles of triangle ACE intersects line BC at points C and G. Prove that $BF = CG$.

by Petru Braica, Romania

G2   Let ABC be a triangle inscribed in circle Γ with center O. Let H be the orthocenter of triangle ABC and let K be the midpoint of OH. Tangent of Γ at B intersects the perpendicular bisector of AC at L. Tangent of Γ at C intersects the perpendicular bisector of AB at M. Prove that AK and LM are perpendicular.

by Michael Sarantis, Greece

G3   Let P be an interior point of triangle ABC. Let a, b, c be the sidelengths of triangle ABC and let p be it's semiperimeter. Find the maximum possible value of

$$\min\left(\frac{PA}{p-a}, \frac{PB}{p-b}, \frac{PC}{p-c}\right)$$

taking into consideration all possible choices of triangle ABC and of point P.

by Elton Bojaxhiu, Albania

G4   A quadrilateral $ABCD$ is inscribed in a circle k where $AB > CD$,and AB is not paralel to CD.Point M is the intersection of diagonals AC and BD, and the perpendicular from M to AB intersects the segment AB at a point E.If EM bisects the angle CED prove that AB is diameter of k.
Proposed by Emil Stoyanov,Bulgaria

G5   Let ABC be an acute triangle with $AB < AC < BC$ and let D be a point on it's extension of BC towards C. Circle c_1, with center A and radius AD, intersects lines AC, AB and CB at points E, F, and G respectively. Circumscribed circle c_2 of triangle AFG intersects again lines FE, BC, GE and DF at points J, H, H' and J' respectively. Circumscribed circle c_3 of triangle ADE intersects again lines FE, BC, GE and DF at points I, K, K' and I' respectively. Prove that the quadrilaterals $HIJK$ and $H'I'J'K'$ are cyclic and the centers of their circumscribed circles coincide.

by Evangelos Psychas, Greece

G6   In a triangle ABC with $AB = AC$, ω is the circumcircle and O its center. Let D be a point on the extension of BA beyond A. The circumcircle ω_1 of triangle OAD intersects the line AC and the circle ω again at points E and G, respectively. Point H is such that $DAEH$ is a parallelogram.

Line EH meets circle ω_1 again at point J. The line through G perpendicular to GB meets ω_1 again at point N and the line through G perpendicular to GJ meets ω again at point L. Prove that the points L, N, H, G lie on a circle.

–   Number Theory

N1   For positive integers m and n, let $d(m,n)$ be the number of distinct primes that divide both m and n. For instance, $d(60, 126) = d(2^2 \cdot 3 \cdot 5, 2 \cdot 3^2 \cdot 7) = 2$. Does there exist a sequence (a_n) of positive integers such that:
- $a_1 \geq 2018^{2018}$;
- $a_m \leq a_n$ whenever $m \leq n$;
- $d(m,n) = d(a_m, a_n)$ for all positive integers $m \neq n$?

(Dominic Yeo, United Kingdom)

N2   Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that

$$n! + f(m)! \,|\, f(n)! + f(m!)$$

for all $m, n \in \mathbb{N}$
Proposed by Valmir Krasniqi and Dorlir Ahmeti, Albania

N3   Find all primes p and q such that $3p^{q-1} + 1$ divides $11^p + 17^p$

Proposed by Stanislav Dimitrov,Bulgaria

N4   Let $P(x) = a_d x^d + \cdots + a_1 x + a_0$ be a non-constant polynomial with non-negative integer coefficients having d rational roots.Prove that

$$\text{lcm}\,(P(m), P(m+1), \ldots, P(n)) \geq m\binom{n}{m}$$

for all $n > m$

(Navid Safaei, Iran)

N5   Let x, y be positive integers. If for each positive integer n we have that

$$(ny)^2 + 1 \mid x^{\varphi(n)} - 1.$$

Prove that $x = 1$.

(Silouanos Brazitikos, Greece)