Balkan MO Shortlist 2016

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– Algebra

A1 Let a, b, c be positive real numbers.
Prove that $\sqrt{a^3b+a^3c}+\sqrt{b^3c+b^3a}+\sqrt{c^3a+c^3b}\geq \frac{4}{3}(ab+bc+ca)$

A2 For all $x, y, z > 0$ satisfying $\frac{x}{yz}+\frac{y}{zx}+\frac{z}{xy}\leq x+y+z$, prove that

$$\frac{1}{x^2+y+z}+\frac{1}{y^2+z+x}+\frac{1}{z^2+x+y}\leq 1$$

A3 Find all injective functions $f:\mathbb{R}\rightarrow\mathbb{R}$ such that for every real number x and every positive integer n,

$$\left|\sum_{i=1}^{n} i\left(f(x+i+1)-f(f(x+i))\right)\right| < 2016$$

(Macedonia)

A4 The positive real numbers a, b, c satisfy the equality $a+b+c=1$. For every natural number n find the minimal possible value of the expression

$$E=\frac{a^{-n}+b}{1-a}+\frac{b^{-n}+c}{1-b}+\frac{c^{-n}+a}{1-c}$$

A5 Let a, b, c and d be real numbers such that $a+b+c+d=2$ and $ab+bc+cd+da+ac+bd=0$.
Find the minimum value and the maximum value of the product $abcd$.

A6 Prove that there is no function from positive real numbers to itself, $f:(0,+\infty)\rightarrow(0,+\infty)$ such that: $f(f(x)+y)=f(x)+3x+yf(y)$,for every $x, y\in(0,+\infty)$

by Greece, Athanasios Kontogeorgis (aka socrates)

A7 Find all integers $n\geq 2$ for which there exist the real numbers $a_k, 1\leq k\leq n$, which are satisfying the following conditions:

$$\sum_{k=1}^{n} a_k=0, \sum_{k=1}^{n} a_k^2=1 \text{ and } \sqrt{n}\cdot\left(\sum_{k=1}^{n} a_k^3\right)=2(b\sqrt{n}-1), \text{ where } b=\max_{1\leq k\leq n}\{a_k\}.$$

A8 Find all functions $f : Z \to Z$ for which $f(g(n)) - g(f(n))$ is independent on n for any $g : Z \to Z$.

– Combinatorics

C1 Let positive integers K and d be given. Prove that there exists a positive integer n and a sequence of K positive integers $b_1, b_2, ..., b_K$ such that the number n is a d-digit palindrome in all number bases $b_1, b_2, ..., b_K$.

C2 There are 2016 costumers who entered a shop on a particular day. Every customer entered the shop exactly once. (i.e. the customer entered the shop, stayed there for some time and then left the shop without returning back.)
Find the maximal k such that the following holds:
There are k customers such that either all of them were in the shop at a speci c time instance or no two of them were both in the shop at any time instance.

C3 The plane is divided into squares by two sets of parallel lines, forming an infinite grid. Each unit square is coloured with one of 1201 colours so that no rectangle with perimeter 100 contains two squares of the same colour. Show that no rectangle of size 1×1201 or 1201×1 contains two squares of the same colour.

Note: Any rectangle is assumed here to have sides contained in the lines of the grid.

(Bulgaria - Nikolay Beluhov)

– Geometry

G1 Let $ABCD$ be a quadrilateral ,circumscribed about a circle. Let M be a point on the side AB. Let I_1,I_2 and I_3 be the incentres of triangles AMD, CMD and BMC respectively. Prove that $I_1I_2I_3M$ is circumscribed.

G2 Let $ABCD$ be a cyclic quadrilateral with $AB < CD$. The diagonals intersect at the point F and lines AD and BC intersect at the point E. Let K and L be the orthogonal projections of F onto lines AD and BC respectively, and let M, S and T be the midpoints of EF, CF and DF respectively. Prove that the second intersection point of the circumcircles of triangles MKT and MLS lies on the segment CD.

(Greece - Silouanos Brazitikos)

G3 Given that ABC is a triangle where $AB < AC$. On the half-lines BA and CA we take points F and E respectively such that $BF = CE = BC$. Let M, N and H be the mid-points of the segments BF, CE and BC respectively and K and O be the circumcenters of the triangles ABC and MNH respectively. We assume that OK cuts BE and HN at the points A_1 and B_1 respectively and that C_1 is the point of intersection of HN and FE. If the parallel line from A_1

to OC_1 cuts the line FE at D and the perpendicular from A_1 to the line DB_1 cuts FE at the point M_1, prove that E is the orthocenter of the triangle A_1OM_1.

– Number Theory

N1 Find all natural numbers n for which $1^{\phi(n)} + 2^{\phi(n)} + ... + n^{\phi(n)}$ is coprime with n.

N2 Find all odd natural numbers n such that $d(n)$ is the largest divisor of the number n different from n.
($d(n)$ is the number of divisors of the number n including 1 and n).

N3 Find all the integer solutions (x, y, z) of the equation $(x + y + z)^5 = 80xyz(x^2 + y^2 + z^2)$,

N4 Find all monic polynomials f with integer coefficients satisfying the following condition: there exists a positive integer N such that p divides $2(f(p)!) + 1$ for every prime $p > N$ for which $f(p)$ is a positive integer.

Note: A monic polynomial has a leading coefficient equal to 1.

(Greece - Panagiotis Lolas and Silouanos Brazitikos)

N5 A positive integer is called *downhill* if the digits in its decimal representation form a nonstrictly decreasing sequence from left to right. Suppose that a polynomial $P(x)$ with rational coefficients takes on an integer value for each downhill positive integer x. Is it necessarily true that $P(x)$ takes on an integer value for each integer x?