Balkan MO Shortlist 2017

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by parmenides51, florin_rotaru, dangerousliri, Snakes, IstekOlympiadTeam, Borbas, sqing, Ferid.—.

– Algebra

A1 Problem Shortlist BMO 2017

Let a,b,c, be positive real numbers such that $abc = 1$. Prove that

$$\frac{1}{a^5+b^5+c^2}+\frac{1}{b^5+c^5+a^2}+\frac{1}{c^5+b^5+b^2} \leq 1.$$

A2 Consider the sequence of rational numbers defined by $x_1 = \frac{4}{3}$ and $x_{n+1} = \frac{x_n^2}{x_n^2-x_n+1}$, $n \geq 1$.
Show that the numerator of the lowest term expression of each sum $\sum_{k=1}^{n} x_k$ is a perfect square.

Proposed by Dorlir Ahmeti, Albania

A3 Let $\mathbb{N}$ denote the set of positive integers. Find all functions $f : \mathbb{N} \longrightarrow \mathbb{N}$ such that

$$n + f(m) \mid f(n) + nf(m)$$

for all $m, n \in \mathbb{N}$

Proposed by Dorlir Ahmeti, Albania

A4 Let $M = \{(a, b, c) \in R^3 : 0 < a, b, c < \frac{1}{2}$ with $a + b + c = 1\}$ and $f : M \to R$ given as

$$f(a,b,c) = 4\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) - \frac{1}{abc}$$

Find the best (real) bounds α and β such that $f(M) = \{f(a, b, c) : (a, b, c) \in M\} \subseteq [\alpha, \beta]$ and determine whether any of them is achievable.

A5 Consider integers $m \geq 2$ and $n \geq 1$.
Show that there is a polynomial $P(x)$ of degree equal to n with integer coefficients such that $P(0), P(1), ..., P(n)$ are all perfect powers of m .

A6 Find all functions $f:\mathbb{R}\rightarrow\mathbb{R}$ such that

$$f(x+yf(x^2))=f(x)+xf(xy)$$

for all real numbers x and y.

– Combinatorics

C1 A grasshopper is sitting at an integer point in the Euclidean plane. Each second it jumps to another integer point in such a way that the jump vector is constant. A hunter that knows neither the starting point of the grasshopper nor the jump vector (but knows that the jump vector for each second is constant) wants to catch the grasshopper. Each second the hunter can choose one integer point in the plane and, if the grasshopper is there, he catches it. Can the hunter always catch the grasshopper in a finite amount of time?

C2 Let n,a,b,c be natural numbers. Every point on the coordinate plane with integer coordinates is colored in one of n colors. Prove there exists c triangles whose vertices are colored in the same color, which are pairwise congruent, and which have a side whose lenght is divisible by a and a side whose lenght is divisible by b.

C3 In the plane, there are n points ($n\geq 4$) where no 3 of them are collinear. Let $A(n)$ be the number of parallelograms whose vertices are those points with area 1. Prove the following inequality: $A(n)\leq \frac{n^2-3n}{4}$ for all $n\geq 4$

C4 For any set of points $A_1,A_2,...,A_n$ on the plane, one defines $r(A_1,A_2,...,A_n)$ as the radius of the smallest circle that contains all of these points. Prove that if $n\geq 3$, there are indices i,j,k such that $r(A_1,A_2,...,A_n)=r(A_i,A_j,A_k)$

C5 On a circular table sit $n>2$ students. First, each student has just one candy. At each step, each student chooses one of the following actions:

(A) Gives a candy to the student sitting on his left or to the student sitting on his right.

(B) Separates all its candies in two, possibly empty, sets and gives one set to the student sitting on his left and the other to the student sitting on his right.

At each step, students perform the actions they have chosen at the same time.

A distribution of candy is called legitimate if it can occur after a finite number of steps.
Find the number of legitimate distributions.

(Two distributions are different if there is a student who has a different number of candy in each of these distributions.)

Forgive my poor English

C6 What is the least positive integer k such that, in every convex 101-gon, the sum of any k diagonals is greater than or equal to the sum of the remaining diagonals?

– Geometry

G1 Let ABC be an acute triangle. Variable points E and F are on sides AC and AB respectively such that $BC^2 = BA \cdot BF + CE \cdot CA$. As E and F vary prove that the circumcircle of AEF passes through a fixed point other than A .

G2 Let ABC be an acute triangle and D a variable point on side AC . Point E is on BD such that $BE = \frac{BC^2 - CD \cdot CA}{BD}$. As D varies on side AC prove that the circumcircle of ADE passes through a fixed point other than A .

G3 Consider an acute-angled triangle ABC with $AB < AC$ and let ω be its circumscribed circle. Let t_B and t_C be the tangents to the circle ω at points B and C, respectively, and let L be their intersection. The straight line passing through the point B and parallel to AC intersects t_C in point D. The straight line passing through the point C and parallel to AB intersects t_B in point E. The circumcircle of the triangle BDC intersects AC in T, where T is located between A and C. The circumcircle of the triangle BEC intersects the line AB (or its extension) in S, where B is located between S and A. Prove that ST, AL, and BC are concurrent.

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G4 The acuteangled triangle ABC with circumcenter O is given. The midpoints of the sides BC, CA and AB are D, E and F respectively. An arbitrary point M on the side BC, different of D, is choosen. The straight lines AM and EF intersects at the point N and the straight line ON cut again the circumscribed circle of the triangle ODM at the point P. Prove that the reflection of the point M with respect to the midpoint of the segment DP belongs on the nine points circle of the triangle ABC.

G5 Let ABC be an acute angled triangle with orthocenter H. centroid G and circumcircle ω. Let D and M respectively be the intersection of lines AH and AG with side BC. Rays MH and DG interect ω again at P and Q respectively. Prove that PD and QM intersect on ω.

G6 Construct outside the acute-angled triangle ABC the isosceles triangles $ABA_B, ABB_A, ACA_C, ACC_A, BC$ and BCC_B, so that

$$AB = AB_A = BA_B, AC = AC_A = CA_C, BC = BC_B = CB_C$$

and

$$\angle BAB_A = \angle ABA_B = \angle CAC_A = \angle ACA_C = \angle BCB_C = \angle CBC_B = a < 90^o$$

.

Prove that the perpendiculars from A to B_AC_A, from B to A_BC_B and from C to A_CB_C are concurrent

G7 Let ABC be an acute triangle with $AB \neq AC$ and circumcircle ω. The angle bisector of BAC intersects BC and ω at D and E respectively. Circle with diameter DE intersects ω again at $F \neq E$. Point P is on AF such that $PB = PC$ and X and Y are feet of perpendiculars from P to AB and AC respectively. Let H and H' be the orthocenters of ABC and AXY respectively. AH meets ω again at Q . If AH' and HH' intersect the circle with diameter AH again at points S and T, respectively, prove that the lines AT, HS and FQ are concurrent.

G8 Given an acute triangle ABC ($AC \neq AB$) and let (C) be its circumcircle. The excircle (C_1) corresponding to the vertex A, of center I_a, tangents to the side BC at the point D and to the extensions of the sides AB, AC at the points E, Z respectively. Let I and L are the intersection points of the circles (C) and (C_1), H the orthocenter of the triangle EDZ and N the midpoint of segment EZ. The parallel line through the point l_a to the line HL meets the line HI at the point G. Prove that the perpendicular line (e) through the point N to the line BC and the parallel line (δ) through the point G to the line IL meet each other on the line HI_a.

– Number Theory

N1 Find all ordered pairs of positive integers(x, y) such that:

$$x^3 + y^3 = x^2 + 42xy + y^2.$$

N2 Find all functions $f : Z_{>0} \to Z_{>0}$ such that the number $xf(x) + f^2(y) + 2xf(y)$ is a perfect square for all positive integers x, y.

N3 Prove that for all positive integer n, there is a positive integer m that $7^n | 3^m + 5^m - 1$.

N4 Find all pairs of positive integers (x, y) , such that x^2 is divisible by $2xy^2 - y^3 + 1$.

N5 Given a positive odd integer n, show that the arithmetic mean of fractional parts $\{\frac{k^{2n}}{p}\}, k = 1, ..., \frac{p-1}{2}$ is the same for infinitely many primes p .