South East Mathematical Olympiad 2019

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by Henry_2001, MarkBcc168, mofumofu, YRNG-BCC168

–	Grade 10
Day 1	July 30th

1 Find the largest real number k, such that for any positive real numbers a, b,

$$(a+b)(ab+1)(b+1) \geq kab^2$$

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2 Two circles Γ_1 and Γ_2 intersect at A, B. Points C, D lie on Γ_1, points E, F lie on Γ_2 such that A, B lies on segments CE, DF respectively and segments CE, DF do not intersect. Let CF meet Γ_1, Γ_2 again at K, L respectively, and DE meet Γ_1, Γ_2 at M, N respectively. If the circumcircles of $\triangle ALM$ and $\triangle BKN$ are tangent, prove that the radii of these two circles are equal.

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3 Let $f : \mathbb{N} \to \mathbb{N}$ be a function such that $f(ab)$ divides $\max\{f(a), b\}$ for any positive integers a, b. Must there exist infinitely many positive integers k such that $f(k) = 1$?

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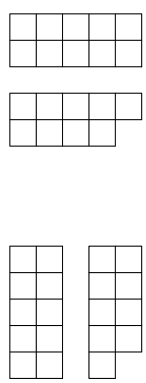
4 As the figure is shown, place a 2×5 grid table in horizontal or vertical direction, and then remove arbitrary one 1×1 square on its four corners. The eight different shapes consisting of the remaining nine small squares are called *banners*.



Here is a fixed 9×18 grid table. Find the number of ways to cover the grid table completely with 18 *banners*.

Day 2 July 31st

5 Let $S = \{1928, 1929, 1930, \cdots, 1949\}$. We call one of Ss subset M is a *red* subset, if the sum of any two different elements of M isnt divided by 4. Let x, y be the number of the *red* subsets of S with 4 and 5 elements,respectively. Determine which of x, y is greater and prove that.

6 Let a, b, c be the lengths of the sides of a given triangle.If positive reals x, y, z satisfy $x+y+z = 1$, find the maximum of $axy + byz + czx$.

7 Let $ABCD$ be a given convex quadrilateral in a plane. Prove that there exist a line with four different points P, Q, R, S on it and a square $ABCD$ such that P lies on both line AB and AB, Q lies on both line BC and BC, R lies on both line CD and CD, S lies on both line DA and DA.

8 For positive integer $x > 1$, define set S_x as

$$S_x = \{p^\alpha | p \text{ is one of the prime divisor of } x, \alpha \in \mathbb{N}, p^\alpha | x, \alpha \equiv v_p(x) (\text{mod} 2)\},$$

where $v_p(n)$ is the power of prime divisor p in positive integer n. Let $f(x)$ be the sum of all the elements of S_x when $x > 1$, and $f(1) = 1$.
Let m be a given positive integer, and the sequence $a_1, a_2, \cdots, a_n, \cdots$ satisfy that for any positive integer $n > m$, $a_{n+1} = \max\{f(a_n), f(a_{n-1} + 1), \cdots, f(a_{n-m} + m)\}$. Prove that
(1)there exists constant $A, B (0 < A < 1)$, such that when positive integer x has at least two different prime divisors, $f(x) < Ax + B$ holds;
(2)there exists positive integer Q, such that for any positive integer $n, a_n < Q$.

– Grade 11

Day 1 July 30th

1 Let $[a]$ represent the largest integer less than or equal to a, for any real number a. Let $\{a\} = a - [a]$.

Are there positive integers m, n and $n+1$ real numbers $x_0, x_1, \ldots, x_n$ such that $x_0 = 428$, $x_n = 1928$, $\frac{x_{k+1}}{10} = \left[\frac{x_k}{10}\right] + m + \left\{\frac{x_k}{5}\right\}$ holds?

Justify your answer.

2 $ABCD$ is a parallelogram with $\angle BAD \neq 90$. Circle centered at A radius BA denoted as ω_1 intersects the extended side of AB, CB at points E, F respectively. Suppose the circle centered at D with radius DA, denoted as ω_2, intersects AD, CD at points M, N respectively. Suppose EN, FM intersects at G, AG intersects ME at point T. MF intersects ω_1 at $Q \neq F$, and EN intersects ω_2 at $P \neq N$. Prove that G, P, T, Q concyclic.

3 n symbols line up in a row, numbered as $1, 2, ..., n$ from left to right. Delete every symbol with squared numbers. Renumber the rest from left to right. Repeat the process until all n symbols are deleted. Let $f(n)$ be the initial number of the last symbol deleted. Find $f(n)$ in terms of n and find $f(2019)$.

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4 Let X be a 5×5 matrix with each entry be 0 or 1. Let $x_{i,j}$ be the (i,j)-th entry of X ($i,j=1,2,\dots,5$). Consider all the 24 ordered sequence in the rows, columns and diagonals of X in the following:

$$\begin{aligned}&(x_{i,1},x_{i,2},\dots,x_{i,5}),\ (x_{i,5},x_{i,4},\dots,x_{i,1}),\ (i=1,2,\dots,5)\\&(x_{1,j},x_{2,j},\dots,x_{5,j}),\ (x_{5,j},x_{4,j},\dots,x_{1,j}),\ (j=1,2,\dots,5)\\&(x_{1,1},x_{2,2},\dots,x_{5,5}),\ (x_{5,5},x_{4,4},\dots,x_{1,1})\\&(x_{1,5},x_{2,4},\dots,x_{5,1}),\ (x_{5,1},x_{4,2},\dots,x_{1,5})\end{aligned}$$

Suppose that all of the sequences are different. Find all the possible values of the sum of all entries in X.

Day 2 July 31st

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5 For positive integer n, define a_n as the number of the triangles with integer length of every side and the length of the longest side being $2n$.
(1) Find a_n in terms of n;
(2)If the sequence $\{b_n\}$ satisfying for any positive integer n, $\sum_{k=1}^n(-1)^{n-k}\binom{n}{k}b_k=a_n$. Find the number of positive integer n satisfying that $b_n\leq 2019a_n$.

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6 In $\triangle ABC$, $AB>AC$, the bisectors of $\angle ABC, \angle ACB$ meet sides AC, AB at D, E respectively. The tangent at A to the circumcircle of $\triangle ABC$ intersects ED extended at P. Suppose that $AP=BC$. Prove that $BD\parallel CP$.

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7 Amy and Bob choose numbers from $0,1,2,\cdots,81$ in turn and Amy choose the number first. Every time the one who choose number chooses one number from the remaining numbers. When all 82 numbers are chosen, let A be the sum of all the numbers Amy chooses, and let B be the sum of all the numbers Bob chooses. During the process, Amy tries to make $\gcd(A,B)$ as great as possible, and Bob tries to make $\gcd(A,B)$ as little as possible. Suppose Amy and Bob take the best strategy of each one, respectively, determine $\gcd(A,B)$ when all 82 numbers are chosen.

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8 For positive integer $x>1$, define set S_x as

$$S_x=\{p^\alpha|p \text{ is one of the prime divisor of } x, \alpha\in\mathbb{N}, p^\alpha|x, \alpha\equiv v_p(x)(\text{mod}2)\},$$

where $v_p(n)$ is the power of prime divisor p in positive integer n. Let $f(x)$ be the sum of all the elements of S_x when $x>1$, and $f(1)=1$.

Let m be a given positive integer, and the sequence $a_1, a_2, \cdots, a_n, \cdots$ satisfy that for any positive integer $n > m$, $a_{n+1} = \max\{f(a_n), f(a_{n-1}+1), \cdots, f(a_{n-m}+m)\}$. Prove that
(1)there exists constant $A, B (0 < A < 1)$, such that when positive integer x has at least two different prime divisors, $f(x) < Ax + B$ holds;
(2)there exists positive integer N, l, such that for any positive integer $n \geq N, a_{n+l} = a_n$ holds.