IMC 2019

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by Tintarn, ThE-dArK-lOrD, lvl3BackPack, math90

– Day 1

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1 Evaluate the product

$$\prod_{n=3}^{\infty} \frac{(n^3+3n)^2}{n^6-64}.$$

Proposed by Orif Ibrogimov, ETH Zurich and National University of Uzbekistan and Karen Keryan, Yerevan State University and American University of Armenia, Yerevan

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2 A four-digit number $YEAR$ is called *very good* if the system

$$\begin{aligned} Yx+Ey+Az+Rw&=Y\\ Rx+Yy+Ez+Aw&=E\\ Ax+Ry+Yz+Ew&=A\\ Ex+Ay+Rz+Yw&=R \end{aligned}$$

of linear equations in the variables x, y, z and w has at least two solutions. Find all very good $YEAR$s in the 21st century.

(The 21st century starts in 2001 and ends in 2100.)

Proposed by Tom Brta, Charles University, Prague

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3 Let $f:(-1,1)\to\mathbb{R}$ be a twice differentiable function such that

$$2f(x)+xf''(x)\geqslant 1 \quad \text{for } x\in(-1,1).$$

Prove that

$$\int_{-1}^{1} xf(x)dx \geqslant \frac{1}{3}.$$

Proposed by Orif Ibrogimov, ETH Zurich and National University of Uzbekistan and Karim Rakhimov, Scuola Normale Superiore and National University of Uzbekistan

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4 Let $(n+3)a_{n+2}=(6n+9)a_{n+1}-na_n$ and $a_0=1$ and $a_1=2$ prove that all the terms of the sequence are integers

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5 Determine whether there exist an odd positive integer n and $n \times n$ matrices A and B with integer entries, that satisfy the following conditions:

-$\det(B) = 1$;
-$AB = BA$;
-$A^4 + 4A^2B^2 + 16B^4 = 2019I$.

(Here I denotes the $n \times n$ identity matrix.)

Proposed by Orif Ibrogimov, ETH Zurich and National University of Uzbekistan

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– Day 2

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6 Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuous functions such that g is differentiable. Assume that $(f(0) - g'(0))(g'(1) - f(1)) > 0$. Show that there exists a point $c \in (0, 1)$ such that $f(c) = g'(c)$.

Proposed by Fereshteh Malek, K. N. Toosi University of Technology

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7 Let $C = \{4, 6, 8, 9, 10, \ldots\}$ be the set of composite positive integers. For each $n \in C$ let a_n be the smallest positive integer k such that $k!$ is divisible by n. Determine whether the following series converges:

$$\sum_{n \in C} \left(\frac{a_n}{n}\right)^n.$$

Proposed by Orif Ibrogimov, ETH Zurich and National University of Uzbekistan

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8 Let $x_1, \ldots, x_n$ be real numbers. For any set $I \subset \{1, 2, , n\}$ let $s(I) = \sum_{i \in I} x_i$. Assume that the function $I \to s(I)$ takes on at least 1.8^n values where I runs over all 2^n subsets of $\{1, 2, , n\}$. Prove that the number of sets $I \subset \{1, 2, , n\}$ for which $s(I) = 2019$ does not exceed 1.7^n.

Proposed by Fedor Part and Fedor Petrov, St. Petersburg State University

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9 Determine all positive integers n for which there exist $n \times n$ real invertible matrices A and B that satisfy $AB - BA = B^2A$.

Proposed by Karen Keryan, Yerevan State University & American University of Armenia, Yerevan

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10 2019 points are chosen at random, independently, and distributed uniformly in the unit disc $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$. Let C be the convex hull of the chosen points. Which probability is larger: that C is a polygon with three vertices, or a polygon with four vertices?

Proposed by Fedor Petrov, St. Petersburg State University
