Moscow Mathematical Olympiad 1952

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by parmenides51

–	tour 1
208	The circle is inscribed in $\vartriangle ABC$. Let L, M, N be the tangent points of the circle with sides AB, AC, BC, respectively. Prove that $\angle MLN$ is always an acute angle.
209	Prove the identity: a) $(ax+by+cz)^2+(bx+cy+az)^2+(cx+ay+bz)^2 = (cx+by+az)^2+(bx+ay+cz)^2+(ax+cy+bz)^2$ b) $(ax + by + cz + du)^2 + (bx + cy + dz + au)^2 + (cx + dy + az + bu)^2 + (dx + ay + bz + cu)^2 = (dx + cy + bz + au)^2 + (cx + by + az + du)^2 + (bx + ay + dz + cu)^2 + (ax + dy + cz + bu)^2$.
210	Prove that if all faces of a parallelepiped are equal parallelograms, they are rhombuses.
211	Two men, A and B, set out from town M to town N, which is 15 km away. Their walking speed is 6 km/hr. They also have a bicycle which they can ride at 15 km/hr. Both A and B start simultaneously, A walking and B riding a bicycle until B meets a pedestrian girl, C, going from N to M. Then B lends his bicycle to C and proceeds on foot; C rides the bicycle until she meets A and gives A the bicycle which A rides until he reaches N. The speed of C is the same as that of A and B. The time spent by A and B on their trip is measured from the moment they started from M until the arrival of the last of them at N. a) When should the girl C leave N for A and B to arrive simultaneously in N? b) When should C leave N to minimize this time?
212	Prove that if the orthocenter divides all heights of a triangle in the same proportion, the triangle is equilateral.
213	Given a geometric progression whose denominator q is an integer not equal to 0 or -1, prove that the sum of two or more terms in this progression cannot equal any other term in it.
214	Prove that if $\|x\| < 1$ and $\|y\| < 1$, then $\left\|\frac{x-y}{1-xy}\right\| < 1$.
215	$\vartriangle ABC$ is divided by a straight line BD into two triangles. Prove that the sum of the radii of circles inscribed in triangles ABD and DBC is greater than the radius of the circle inscribed in $\vartriangle ABC$.
216	A sequence of integers is constructed as follows: a_1 is an arbitrary three-digit number, a_2 is the sum of squares of the digits of a_1, a_3 is the sum of squares of the digits of a_2, etc. Prove that either 1 or 4 must occur in the sequence $a_1, a_2, a_3, ...$

217 Given three skew lines. Prove that they are pair-wise perpendicular to their pair-wise perpendiculars.

218 How $arc\sin(\cos(arc\sin x))$ and $arc\cos(\sin(arc\cos x))$ are related with each other?

219 Prove that $(1-x)^n + (1+x)^n < 2^n$ for an integer $n \geq 2$ and $|x| < 1$.

220 A sphere with center at O is inscribed in a trihedral angle with vertex S. Prove that the plane passing through the three tangent points is perpendicular to OS.

221 Prove that if for any positive p all roots of the equation $ax^2 + bx + c + p = 0$ are real and positive then $a = 0$.

– tour 2

222 a) Solve the system of equations $\begin{cases} 1 - x_1x_2 = 0 \\ 1 - x_2x_3 = 0 \\ ... \\ 1 - x_{14}x_{15} = 0 \\ 1 - x_{15}x_1 = 0 \end{cases}$

b) Solve the system of equations $\begin{cases} 1 - x_1x_2 = 0 \\ 1 - x_2x_3 = 0 \\ ... \\ 1 - x_{n-1}x_n = 0 \\ 1 - x_nx_1 = 0 \end{cases}$

How does the solution vary for distinct values of n?

223 In a convex quadrilateral $ABCD$, let $AB + CD = BC + AD$. Prove that the circle inscribed in ABC is tangent to the circle inscribed in ACD.

224- You are given a segment AB. Find the locus of the vertices C of acute-angled triangles ABC.

224 a) Prove that if the square of a number begins with $0.\underbrace{9...9}_{100}$, then the number itself begins with $0.\underbrace{9...9}_{100}$.

b) Calculate $\sqrt{0.9...9}$ (60 nines) to 60 decimal places

225 From a point C, tangents CA and CB are drawn to a circle O. From an arbitrary point N on the circle, perpendiculars ND, NE, NF are drawn on AB, CA and CB, respectively. Prove that the length of ND is the mean proportional of the lengths of NE and NF.

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226 Seven chips are numbered $1, 2, 3, 4, 5, 6, 7$. Prove that none of the seven-digit numbers formed by these chips is divisible by any other of these seven-digit numbers.

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227 99 straight lines divide a plane into n parts. Find all possible values of n less than 199.

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228 How to arrange three right circular cylinders of diameter $a/2$ and height a into an empty cube with side a so that the cylinders could not change position inside the cube? Each cylinder can, however, rotate about its axis of symmetry.

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229 In an isosceles triangle $\triangle ABC, \angle ABC = 20^o$ and $BC = AB$. Points P and Q are chosen on sides BC and AB, respectively, so that $\angle PAC = 50^o$ and $\angle QCA = 60^o$. Prove that $\angle PQC = 30^o$.

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230 200 soldiers occupy in a rectangle (military call it a square and educated military a carree): 20 men (per row) times 10 men (per column).
In each row, we consider the tallest man (if some are of equal height, choose any of them) and of the 10 men considered we select the shortest (if some are of equal height, choose any of them). Call him A.
Next the soldiers assume their initial positions and in each column the shortest soldier is selected, of these 20, the tallest is chosen. Call him B.
Two colonels bet on which of the two soldiers chosen by these two distinct procedures is taller: A or B.
Which colonel wins the bet?

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231 Prove that for arbitrary fixed $a_1, a_2, .., a_{31}$ the sum $\cos 32x + a_{31} \cos 31x + ... + a_2 cos 2x + a_1 \cos x$ can take both positive and negative values as x varies.

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232 Prove that for any integer a the polynomial $3x^{2n} + ax^n + 2$ cannot be divided by $2x^{2m} + ax^m + 3$ without a remainder.
