Moscow Mathematical Olympiad 1953

www.artofproblemsolving.com/community/c918097

by parmenides51

–	tour 1
233	Prove that the sum of angles at the longer base of a trapezoid is less than the sum of angles at the shorter base.
234	Find the smallest number of the form $1...1$ in its decimal expression which is divisible by $\underbrace{3...3}_{100}$.
235	Divide a segment in halves using a right triangle. (With a right triangle one can draw straight lines and erect perpendiculars but cannot draw perpendiculars.)
236	Prove that $n^2 + 8n + 15$ is not divisible by $n + 4$ for any positive integer n.
237	Three circles are pair-wise tangent to each other. Prove that the circle passing through the three tangent points is perpendicular to each of the initial three circles.
238	Prove that if in the following fraction we have n radicals in the numerator and $n - 1$ in the denominator, then $\frac{2-\sqrt{2+\sqrt{2+...+\sqrt{2}}}}{2-\sqrt{2+\sqrt{2+...+\sqrt{2}}}} > \frac{1}{4}$
239	On the plane find the locus of points whose coordinates satisfy $sin(x + y) = 0$.
240	Let AB and A_1B_1 be two skew segments, O and O_1 their respective midpoints. Prove that OO_1 is shorter than a half sum of AA_1 and BB_1.
241	Prove that the polynomial $x^{200}y^{200} + 1$ cannot be represented in the form $f(x)g(y)$, where f and g are polynomials of only x and y, respectively.
242	Let A be a vertex of a regular star-shaped pentagon, the angle at A being less than 180^o and the broken line $AA_1BB_1CC_1DD_1EE_1$ being its contour. Lines AB and DE meet at F. Prove that polygon $ABB_1CC_1DED_1$ has the same area as the quadrilateral AD_1EF.
243	Given a right circular cone and a point A. Find the set of vertices of cones equal to the given one, with axes parallel to that of the given one, and with A inside them.

– tour 2

244 Prove that $gcd(a+b, lcm(a,b)) = gcd(a,b)$ for any a, b.

245 A quadrilateral is circumscribed around a circle. Its diagonals intersect at the center of the circle. Prove that the quadrilateral is a rhombus.

246 a) On a plane, 11 gears are arranged so that the teeth of the first gear mesh with the teeth of the second gear, the teeth of the second gear with those of the third gear, etc., and the teeth of the last gear mesh with those of the first gear. Can the gears rotate?
b) On a plane, n gears are arranged so that the teeth of the first gear mesh with the teeth of the second gear, the teeth of the second gear with those of the third gear, etc., and the teeth of the last gear mesh with those of the first gear. Can the gears rotate?

247 Inside a convex 1000-gon, 500 points are selected so that no three of the 1500 points — the ones selected and the vertices of the polygon — lie on the same straight line. This 1000-gon is then divided into triangles so that all 1500 points are vertices of the triangles, and so that these triangles have no other vertices.
How many triangles will there be?

248 a) Solve the system $\begin{cases} x_1+2x_2+2x_3+2x_4+2x_5=1 \\ x_1+3x_2+4x_3+4x_4+4x_5=2 \\ x_1+3x_2+5x_3+6x_4+6x_5=3 \\ x_1+3x_2+5x_3+7x_4+8x_5=4 \\ x_1+3x_2+5x_3+7x_4+9x_5=5 \end{cases}$

b) Solve the system $\begin{cases} x_1+2x_2+2x_3+2x_4+2x_5+...+2x_{100}=1 \\ x_1+3x_2+4x_3+4x_4+4x_5+...+4x_{100}=2 \\ x_1+3x_2+5x_3+6x_4+6x_5+...+6x_{100}=3 \\ x_1+3x_2+5x_3+7x_4+8x_5+...+8x_{100}=4 \\ ... \\ x_1+3x_2+5x_3+7x_4+9x_5+...+199x_{100}=100 \end{cases}$

249 Let a, b, c, d be the lengths of consecutive sides of a quadrilateral, and S its area.
Prove that $S \le \frac{(a+b)(c+d)}{4}$

250 Somebody wrote 1953 digits on a circle. The 1953-digit number obtained by reading these figures clockwise, beginning at a certain point, is divisible by 27. Prove that if one begins reading the figures at any other place, one gets another 1953-digit number also divisible by 27.

251 On a circle, distinct points $A_1, ..., A_{16}$ are chosen. Consider all possible convex polygons all of whose vertices are among $A_1, ..., A_{16}$. These polygons are divided into 2 groups, the first group comprising all polygons with A_1 as a vertex, the second group comprising the remaining polygons. Which group is more numerous?

---

252 Given triangle $\triangle A_1A_2A_3$ and a straight line ℓ outside it. The angles between the lines A_1A_2 and A_2A_3, A_1A_2 and A_2A_3, A_2A_3 and A_3A_1 are equal to a_3, a_1 and a_2, respectively. The straight lines are drawn through points A_1, A_2, A_3 forming with ℓ angles of $\pi - a_1, \pi - a_2, \pi - a_3$, respectively. All angles are counted in the same direction from ℓ . Prove that these new lines meet at one point.

---

253 Given the equations
(1) $ax^2 + bx + c = 0$
(2)$-ax^2 + bx + c = 0$
prove that if x_1 and x_2 are some roots of equations (1) and (2), respectively,
then there is a root x_3 of the equation $\frac{a}{2}x^2 + bx + c = 0$
such that either $x_1 \le x_3 \le x_2$ or $x_1 \ge x_3 \ge x_2$.

---

254 Given a 101×200 sheet of graph paper, we start moving from a corner square in the direction of the square's diagonal (not the sheet's diagonal) to the border of the sheet, then change direction obeying the laws of light's reflection. Will we ever reach a corner square?
`https://cdn.artofproblemsolving.com/attachments/b/8/4ec2f4583f406feda004c7fb4f11a424c9b9a`
`png`

---

255 Divide a cube into three equal pyramids.

---

256 Find roots of the equation $1 - \frac{x}{1} + \frac{x(x-1)}{2!} - ... + \frac{(-1)^n x(x-1)...(x-n+1)}{n!} = 0.$

---

257 Let $x_0 = 10^9$, $x_n = \frac{x_{n-1}^2+2}{2x_{n-1}}$ for $n > 0$. Prove that $0 < x_{36} - \sqrt{2} < 10^{-9}$.

---

258 A knight stands on an infinite chess board. Find all places it can reach in exactly $2n$ moves.
