Moscow Mathematical Olympiad 1954

www.artofproblemsolving.com/community/c918098

by parmenides51

–	tour 1
259	A regular star-shaped hexagon is split into 4 parts. Construct from them a convex polygon.
260	Given two convex polygons, $A_1A_2...A_n$ and $B_1B_2...B_n$ such that $A_1A_2 = B_1B_2$, $A_2A_3 = B_2B_3$,..., $A_nA_1 = B_nB_1$ and $n - 3$ angles of one polygon are equal to the respective angles of the other. Find whether these polygons are equal.
261	Find a four-digit number whose division by two given distinct one-digit numbers goes along the following lines: `https://cdn.artofproblemsolving.com/attachments/2/a/e1d3c68ec52e11ad59de755c3dbdc2cf54a81 png`
262	Are there integers m and n such that $m^2 + 1954 = n^2$?
263	Define the maximal value of the ratio of a three-digit number to the sum of its digits.
264	* Cut out of a 3×3 square an unfolding of the cube with edge 1.
265	From an arbitrary point O inside a convex n-gon, perpendiculars are drawn on (extensions of the) sides of the n-gon. Along each perpendicular a vector is constructed, starting from O, directed towards the side onto which the perpendicular is drawn, and of length equal to half the length of the corresponding side. Find the sum of these vectors.
266	Find all solutions of the system consisting of 3 equations: $x\left(1-\frac{1}{2^n}\right)+y\left(1-\frac{1}{2^{n+1}}\right)+z\left(1-\frac{1}{2^{n+2}}\right)=0$ for $n = 1, 2, 3$.
267	Prove that if $x_0^4 + a_1x_0^3 + a_2x_0^2 + a_3x_0 + a_4 = 0$ and $4x_0^3 + 3a_1x_0^2 + 2a_2x_0 + a_3 = 0$, then $x^4 + a_1x^3 + a_2x^2 + a_3x + a_4$ is a mutliple of $(x - x_0)^2$.
268	Delete 100 digits from the number $1234567891011...9899100$ so that the remaining number were as big as possible.

269 a) Given 100 numbers $a_1, ..., a_{100}$ such that $\begin{cases} a_1 - 3a_2 + 2a_3 \geq 0, \\ a_2 - 3a_3 + 2a_4 \geq 0, \\ a_3 - 3a_4 + 2a_5 \geq 0, \\ ... \\ a_{99} - 3a_{100} + 2a_1 \geq 0, \\ a_{100} - 3a_1 + 2a_2 \geq 0 \end{cases}$

prove that the numbers are equal.

b) Given numbers $a_1 = 1, ..., a_{100}$ such that $\begin{cases} a_1 - 4a_2 + 3a_3 \geq 0, \\ a_2 - 4a_3 + 3a_4 \geq 0, \\ a_3 - 4a_4 + 3a_5 \geq 0, \\ ... \\ a_{99} - 4a_{100} + 3a_1 \geq 0, \\ a_{100} - 4a_1 + 3a_2 \geq 0 \end{cases}$

Find $a_2, a_3, ..., a_{100}$.

---

270 Consider $\triangle ABC$ and a point S inside it. Let A_1, B_1, C_1 be the intersection points of AS, BS, CS with BC, AC, AB, respectively. Prove that at least in one of the resulting quadrilaterals AB_1SC_1, C_1SA_1B, A both angles at either C_1 and B_1, or C_1 and A_1, or A_1 and B_1 are not acute.

---

271 Do there exist points A, B, C, D in space, such that $AB = CD = 8, AC = BD = 10$, and $AD = BC = 13$?

---

272 Find all real solutions of the equation $x^2 + 2x\sin(xy) + 1 = 0$.

– tour 2

---

273 Given a piece of graph paper with a letter assigned to each vertex of every square such that on every segment connecting two vertices that have the same letter and are on the same line of the mesh, there is at least one vertex with another letter. What is the least number of distinct letters needed to plot such a picture?

---

274 Solve the system $\begin{cases} 10x_1 + 3x_2 + 4x_3 + x_4 + x_5 = 0 \\ 11x_2 + 2x_3 + 2x_4 + 3x_5 + x_6 = 0 \\ 15x_3 + 4x_4 + 5x_5 + 4x_6 + x_7 = 0 \\ 2x_1 + x_2 - 3x_3 + 12x_4 - 3x_5 + x_6 + x_7 = 0 \\ 6x_1 - 5x_2 + 3x_3 - x_4 + 17x_5 + x_6 = 0 \\ 3x_1 + 2x_2 - 3x_3 + 4x_4 + x_5 - 16x_6 + 2x_7 = 0 \\ 4x_1 - 8x_2 + x_3 + x_4 + 3x_5 + 19x_7 = 0 \end{cases}$

---

275 How many axes of symmetry can a heptagon have?

---

276 a) Let $1, 2, 3, 5, 6, 7, 10, .., N$ be all the divisors of $N = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31$ (the product of primes 2 to 31) written in increasing order. Below this series of divisors, write the following series of 1's or -1's: write 1 below any number that factors into an even number of prime factors and below a 1, write -1 below the remaining numbers.
Prove that the sum of the series of 1's and -1's is equal to 0.
b) Let $1, 2, 3, 5, 6, 7, 10, .., N$ be all the divisors of $N = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 37$ (the product of primes 2 to 37) written in increasing order. Below this series of divisors, write the following series of 1's or -1's: write 1 below any number that factors into an even number of prime factors and below a 1, write -1 below the remaining numbers.
Prove that the sum of the series of 1's and -1's is equal to 0.

---

277 The map of a town shows a plane divided into equal equilateral triangles. The sides of these triangles are streets and their vertices are intersections; 6 streets meet at each junction. Two cars start simultaneously in the same direction and at the same speed from points A and B situated on the same street (the same side of a triangle). After any intersection an admissible route for each car is either to proceed in its initial direction or turn through 120^o to the right or to the left. Can these cars meet? (Either prove that these cars won't meet or describe a route by which they will meet.)
https://cdn.artofproblemsolving.com/attachments/2/d/2c934bcd0c7fc3d9dca9cee0b6f015076abbc
png

---

278 A 17×17 square is cut out of a sheet of graph paper. Each cell of this square has one of thenumbers from 1 to 70. Prove that there are 4 distinct squares whose centers A, B, C, D are the vertices of a parallelogramsuch that $AB//CD$, moreover, the sum of the numbers in the squares with centers A and C is equal to that in the squares with centers B and D.

---

279 Given four straight lines, m_1, m_2, m_3, m_4, intersecting at O and numbered clockwise with O as the center of the clock, we draw a line through an arbitrary point A_1 on m_1 parallel to m_4 until the line meets m_2 at A_2. We draw a line through A_2 parallel to m_1 until it meets m_3 at A_3. We also draw a line through A_3 parallel to m_2 until it meets m_4 at A_4. Now, we draw a line throughA_4 parallel to m_3 until it meets m_1 at B. Prove that
a) $OB < \frac{OA_1}{2}$.
b) $OB \le \frac{OA_1}{4}$.
https://cdn.artofproblemsolving.com/attachments/5/f/5ea08453605e02e7e1253fd7c74065a9ffbd8
png

---

280 Rays l_1 and l_2 pass through a point O. Segments OA_1 and OB_1 on l_1, and OA_2 and OB_2 on l_2, are drawn so that $\frac{OA_1}{OA_2} \ne \frac{OB_1}{OB_2}$. Find the set of all intersection points of lines A_1A_2 and B_1B_2 as l_2 rotates around O while l_1 is fixed.

---

281 *. Positive numbers $x_1, x_2, ..., x_{100}$ satisfy the system $\begin{cases} x_1^2 + x_2^2 + ... + x_{100}^2 > 10000 \\ x_1 + x_2 + ... + x_{100} < 300 \end{cases}$
Prove that among these numbers there are three whose sum is greater than 100.

---

282 Given a sequence of numbers $a_1, a_2, ..., a_{15}$, one can always construct a new sequence $b_1, b_2, ..., b_{15}$, where b_i is equal to the number of terms in the sequence $\{a_k\}_{k=1}^{15}$ less than a_i $(i = 1, 2, ..., 15)$.
Is there a sequence $\{a_k\}_{k=1}^{15}$ for which the sequence $\{b_k\}_{k=1}^{15}$ is $1, 0, 3, 6, 9, 4, 7, 2, 5, 8, 8, 5, 10, 13, 13$?

---

283 Consider five segments $AB_1, AB_2, AB_3, AB_4, AB_5$. From each point B_i there can exit either 5 segments or no segments at all, so that the endpoints of any two segments of the resulting graph (system of segments) do not coincide. Can the number of free endpoints of the segments thus constructed be equal to 1001? (A free endpoint is an endpoint from which no segment begins.)

---

284 How many planes of symmetry can a triangular pyramid have?

---

285 The absolute values of all roots of the quadratic equation $x^2 + Ax + B = 0$ and $x^2 + Cx + D = 0$ are less then 1. Prove that so are absolute values of the roots of the quadratic equation $x^2 + \frac{A+C}{2}x + \frac{B+D}{2} = 0$.

---

286 Consider the set of all 10-digit numbers expressible with the help of figures 1 and 2 only. Divide it into two subsets so that the sum of any two numbers of the same subset is a number which is written with not less than two 3's.
