2018 Israel National Olympiad

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by Cuubic

1 n people sit in a circle. Each of them is either a liar (always lies) or a truthteller (always tells the truth). Every person knows exactly who speaks the truth and who lies. In their turn, each person says 'the person two seats to my left is a truthteller'. It is known that there's at least one liar and at least one truthteller in the circle.

- Is it possible that $n = 2017?$
- Is it possible that $n = 5778?$

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2 An *arithmetic sequence* is an infinite sequence of the form $a_n = a_0 + n \cdot d$ with $d \neq 0$.

A *geometric sequence* is an infinite sequence of the form $b_n = b_0 \cdot q^n$ where $q \neq 1, 0, -1$.

- Does every arithmetic sequence of **integers** have an infinite subsequence which is geometric?
- Does every arithmetic sequence of **real numbers** have an infinite subsequence which is geometric?

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3 Determine the minimal and maximal values the expression $\frac{|a+b|+|b+c|+|c+a|}{|a|+|b|+|c|}$ can take, where a, b, c are real numbers.

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4 The three-digit number 999 has a special property: It is divisible by 27, and its digit sum is also divisible by 27. The four-digit number 5778 also has this property, as it is divisible by 27 and its digit sum is also divisible by 27. How many four-digit numbers have this property?

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5 The sequence a_n is defined for any $n \geq 10$ by the following inductive rule:

- $a_{10} = 5778$
- If $a_n = 0$ then $a_{n+1} = 0.$
- If $a_n \neq 0$ then a_{n+1} is the number whose base-$(n+1)$ representation equals the base n representation of the number $a_n - 1$.

For example, $a_{11} = 5 \cdot 11^3 + 7 \cdot 11^2 + 7 \cdot 11^1 + 7 \cdot 11^0 = 7586$ $a_{12} = 5 \cdot 12^3 + 7 \cdot 12^2 + 7 \cdot 12^1 + 6 \cdot 12^0 = 9738$

- Does there exist $n \geq 10$ for which $a_n = 0$?
- Is $a_{1,000,000} = 0$?
- Is $a_{100^{100^{100}}} = 0$?

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6 In the corners of triangle ABC there are three circles with the same radius. Each of them is tangent to two of the triangle's sides. The vertices of triangle MNK lie on different sides of triangle ABC, and each edge of MNK is also tangent to one of the three circles. Likewise, the vertices of triangle PQR lie on different sides of triangle ABC, and each edge of PQR is also tangent to one of the three circles (see picture below). Prove that triangles MNK, PQR have the same inradius.

https://i.imgur.com/bYuBabS.png

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7 A *uniform covering* of the integers $1, 2, ..., n$ is a finite multiset of subsets of $\{1, 2, ..., n\}$, so that each number lies in the same amount of sets from the covering. A covering may contain the same subset multiple times, it must contain at least one subset, and it may contain the empty subset. For example, $(\{1\}, \{1\}, \{2, 3\}, \{3, 4\}, \{2, 4\})$ is a uniform covering of $1, 2, 3, 4$ (every number occurs in two sets). The covering containing only the empty set is also uniform (every number occurs in zero sets).

Given two uniform coverings, we define a new uniform covering, their *sum* (denoted by $\oplus$), by adding the sets from both coverings. For example:

$$(\{1\}, \{1\}, \{2, 3\}, \{3, 4\}, \{2, 4\}) \oplus (\{1\}, \{2\}, \{3\}, \{4\}) = (\{1\}, \{1\}, \{1\}, \{2\}, \{3\}, \{4\}, \{2, 3\}, \{3, 4\}, \{2, 4\})$$

A uniform covering is called *non-composite* if it's not a sum of two uniform coverings.

Prove that for any $n \geq 1$, there are only finitely many non-composite uniform coverings of $1, 2, ..., n$.
