2016 Israel National Olympiad

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by Cuubic

1 Nina and Meir are walking on a 3 km path towards grandma's house. They start walking at the same time from the same point. Meir's speed is 4 km/h and Nina's speed is 3 km/h.

Along the path there are several benches. Whenever Nina or Meir reaches a bench, they sit on it for some time and eat a cookie. Nina always takes t minutes to eat a cookie, and Meir always takes $2t$ minutes to eat a cookie, where t is a positive integer.

It turns out that Nina and Meir reached grandma's house at the same time. How many benches were there? Find all of the options.

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2 We are given a cone with height 6, whose base is a circle with radius $\sqrt{2}$. Inside the cone, there is an inscribed cube: Its bottom face on the base of the cone, and all of its top vertices lie on the cone. What is the length of the cube's edge?

`https://i.imgur.com/AHqHHP6.png`

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3 Denote by $S(n)$ the sum of digits of n. Given a positive integer N, we consider the following process: We take the sum of digits $S(N)$, then take its sum of digits $S(S(N))$, then its sum of digits $S(S(S(N)))$... We continue this until we are left with a one-digit number.

We call the number of times we had to activate $S(\cdot)$ the **depth** of N.

For example, the depth of 49 is 2, since $S(49)=13\rightarrow S(13)=4$, and the depth of 45 is 1, since $S(45)=9$.

- Prove that every positive integer N has a finite depth, that is, at some point of the process we get a one-digit number.
- Define $x(n)$ to be the minimal positive integer with depth n. Find the residue of $x(5776) \mod 6$.
- Find the residue of $x(5776)-x(5708) \mod 2016$.

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4 In the beginning, there is a circle with three points on it. The points are colored (clockwise): Green, blue, red. Jonathan may perform the following actions, as many times as he wants, in any order:

- Choose two adjacent points with different colors, and add a point between them with one of the two colors only.
- Choose two adjacent points with the same color, and add a point between them with any of the three colors.

- Choose three adjacent points, at least two of them having the same color, and delete the middle point.

Can Jonathan reach a state where only three points remain on the circle, colored (clockwise): Blue, green, red?

5 The Fibonacci sequence F_n is defined by $F_1 = F_2 = 1$ and the recurrence relation $F_n = F_{n-1} + F_{n-2}$ for all integers $n \geq 3$.

Let $m, n \geq 1$ be integers. Find the minimal degree d for which there exists a polynomial $f(x) = a_d x^d + a_{d-1} x^{d-1} + \cdots + a_1 x + a_0$, which satisfies $f(k) = F_{m+k}$ for all $k = 0, 1, ..., n$.

6 Points $A_1, A_2, A_3, ..., A_{12}$ are the vertices of a regular polygon in that order. The 12 diagonals $A_1A_6, A_2A_7, A_3A_8, ..., A_{11}A_4, A_{12}A_5$ are marked, as in the picture below. Let X be some point in the plane. From X, we draw perpendicular lines to all 12 marked diagonals. Let $B_1, B_2, B_3, ..., B_{12}$ be the feet of the perpendiculars, so that B_1 lies on A_1A_6, B_2 lies on A_2A_7 and so on.

Evaluate the ratio $\frac{XA_1+XA_2+\cdots+XA_{12}}{B_1B_6+B_2B_7+\cdots+B_{12}B_5}$.
`https://i.imgur.com/DUuwFth.png`

7 Find all functions $f : \mathbb{Z} \to \mathbb{C}$ such that $f(x(2y+1)) = f(x(y+1)) + f(x)f(y)$ holds for any two integers x, y.