2015 Israel National Olympiad

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by Cuubic

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- Find an example of three positive integers a, b, c satisfying $31a + 30b + 28c = 365$.
- Prove that any triplet a, b, c satisfying the above condition, also satisfies $a + b + c = 12$.

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2 A triangle is given whose altitudes' lengths are $\frac{1}{5}, \frac{1}{5}, \frac{1}{8}$. Evaluate the triangle's area.

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3 Prove that the number $\left(\frac{76}{\frac{1}{\sqrt[3]{77}-\sqrt[3]{75}}-\sqrt[3]{5775}}+\frac{1}{\frac{76}{\sqrt[3]{77}+\sqrt[3]{75}}+\sqrt[3]{5775}}\right)^3$ is an integer.

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4 Let k, m, n be positive integers such that n^m is divisible by m^n, and m^k is divisible by k^m.

- Prove that n^k is divisible by k^n.
- Find an example of k, m, n satisfying the above conditions, where all three numbers are distinct and bigger than 1.

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5 Let $ABCD$ be a tetrahedron. Denote by S_1 the inscribed sphere inside it, which is tangent to all four faces. Denote by S_2 the outer escribed sphere outside ABC, tangent to face ABC and to the planes containing faces ABD, ACD, BCD. Let K be the tangency point of S_1 to the face ABC, and let L be the tangency point of S_2 to the face ABC. Let T be the foot of the perpendicular from D to the face ABC.

Prove that L, T, K lie on one line.

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6 Let $n \geq 1$ be a positive integer. n lamps are placed in a line. At minute 0, some lamps are on (maybe all of them). Every minute the state of the lamps changes: A lamp is on at minute $t+1$ if and only if at minute t, exactly one of its neighbors is on (the two lamps at the ends have one neighbor each, all other lamps have two neighbors).

For which values of n can we guarantee that all lamps will be off after some time?

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7 The Fibonacci sequence F_n is defined by $F_0 = 0, F_1 = 1$ and the recurrence relation $F_n = F_{n-1} + F_{n-2}$ for all integers $n \geq 2$. Let $p \geq 3$ be a prime number.

- Prove that $F_{p-1} + F_{p+1} - 1$ is divisible by p.
- Prove that $F_{p^{k+1}-1} + F_{p^{k+1}+1} - \left(F_{p^k-1} + F_{p^k+1}\right)$ is divisible by p^{k+1} for any positive integer k.
