2012 Israel National Olympiad

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by Cuubic

1 In the picture below, the circles are tangent to each other and to the edges of the rectangle. The larger circle's radius equals 1. Determine the area of the rectangle.
https://i.imgur.com/g3GUg4Z.png

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2 In some foreign country, there is a secret object, guarded by seven guards. Each guard has a guarding shift of 7 consecutive hours every day, in fixed hours. There is always at least one guard guarding the secret object at any given time.

Prove that one of the guards can be fired, and there will still be at least one guard guarding at any given time (without changing the schedule of the other guards).

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3 Let a, b, c be real numbers such that $a^3(b+c)+b^3(a+c)+c^3(a+b) = 0$. Prove that $ab+bc+ca \leq 0$.

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4 We are given a 7x7 square board. In each square, one of the diagonals is traced, and then one of the two formed triangles is colored blue. What is the largest area a continuous blue component can have?

(Note: continuous blue component means a set of blue triangles connected via their edges, passing through corners is not permitted)

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5 Find all integer solutions of the equation $a^3 + 3ab^2 + 7b^3 = 2011$.

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6 Let A, B, C, O be points in the plane such that angles $\angle AOB, \angle BOC, \angle COA$ are obtuse. On OA, OB, OC points X, Y, Z respectively are chosen, such that $OX = OY = OZ$. On segments OX, OY, OZ points K, L, M respectively are chosen.

The lines AL and BK intersect at point R, which isn't on XY. The segment XY intersects AL, BK at points R_1, R_2.

The lines BM and CL intersect at point P, which isn't on YZ. The segment YZ intersects BM, CL at points P_1, P_2.

The lines CK and AM intersect at point Q, which isn't on ZX. The segment ZX intersects CK, AM at points Q_1, Q_2.

Suppose that $PP_1 = PP_2$ and $QQ_1 = QQ_2$. Prove that $RR_1 = RR_2$.
