China Girls Math Olympiad 2019

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Day 1 August 12, 2019

1 Let $ABCD$ be a cyclic quadrilateral with circumcircle $\odot O$. The lines tangent to $\odot O$ at A, B intersect at L. M is the midpoint of the segment AB. The line passing through D and parallel to CM intersects $\odot(CDL)$ at F. Line CF intersects DM at K, and intersects $\odot O$ at E (different from point C).
Prove that $EK = DK$.

2 Find integers $a_1, a_2, \cdots, a_{18}$, s.t. $a_1 = 1, a_2 = 2, a_{18} = 2019$, and for all $3 \le k \le 18$, there exists $1 \le i < j < k$ with $a_k = a_i + a_j$.

3 For a sequence, one can perform the following operation: select three adjacent terms a, b, c, and change it into b, c, a. Determine all the possible positive integers $n \ge 3$, such that after finite number of operation, the sequence $1, 2, \cdots, n$ can be changed into $n, n-1, \cdots, 1$ finally.

4 Given parallelogram $OABC$ in the coodinate with O the origin and A, B, C be lattice points. Prove that for all lattice point P in the internal or boundary of $\triangle ABC$, there exists lattice points Q, R(can be the same) in the internal or boundary of $\triangle OAC$ with $\overrightarrow{OP} = \overrightarrow{OQ} + \overrightarrow{OR}$.

Day 2 August 13, 2019

5 Let p be a prime number such that $p \mid (2^{2019} - 1)$. The sequence $a_1, a_2, ..., a_n$ satisfies the following conditions: $a_0 = 2, a_1 = 1, a_{n+1} = a_n + \frac{p^2-1}{4} a_{n-1}$ $(n \ge 1)$. Prove that $p \nmid (a_n + 1)$, for any $n \ge 0$.

6 Let $0 \le x_1 \le x_2 \le \cdots \le x_n \le 1$ $(n \ge 2)$. Prove that

$$\sqrt[n]{x_1 x_2 \cdots x_n} + \sqrt[n]{(1-x_1)(1-x_2)\cdots(1-x_n)} \le \sqrt[n]{1-(x_1-x_n)^2}.$$

7 Let $DFGE$ be a cyclic quadrilateral. Line DF intersects EG at C, and line FE intersects DG at H. J is the midpoint of FG. The line ℓ is the reflection of the line DE in CH, and it intersects line GF at I.
Prove that C, J, H, I are concyclic.

8 For a tournament with 8 vertices, if from any vertex it is impossible to follow a route to return to itself, we call the graph a *good* graph. Otherwise, we call it a *bad* graph. Prove that (1) there

exists a tournament with 8 vertices such that after changing the orientation of any at most 7 edges of the tournament, the graph is always a*bad* graph; (2) for any tournament with 8 vertices, one can change the orientation of at most 8 edges of the tournament to get a *good* graph.
(A tournament is a complete graph with directed edges.)