China Western Mathematical Olympiad 2019

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Day 1 August 13, 2019

1 Determine all the possible positive integer n, such that 3^n+n^2+2019 is a perfect square.

2 Let O, H be the circumcenter and orthocenter of acute triangle ABC with $AB \neq AC$, respectively. Let M be the midpoint of BC and K be the intersection of AM and the circumcircle of $\triangle BHC$, such that M lies between A and K. Let N be the intersection of HK and BC. Show that if $\angle BAM = \angle CAN$, then $AN \perp OH$.

3 Let $S=\{(i,j)|i,j=1,2,\ldots,100\}$ be a set consisting of points on the coordinate plane. Each element of S is colored one of four given colors. A subset T of S is called *colorful* if T consists of exactly 4 points with distinct colors, which are the vertices of a rectangle whose sides are parallel to the coordinate axes. Find the maximum possible number of colorful subsets S can have, among all legitimate coloring patters.

4 Let n be a given integer such that $n\geq 2$. Find the smallest real number λ with the following property: for any real numbers $x_1,x_2,\ldots,x_n\in [0,1]$, there exists integers $\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n\in\{0,1\}$ such that the inequality

$$\left\vert \sum^j_{k=i} (\varepsilon_k-x_k)\right\vert\leq \lambda$$

holds for all pairs of integers (i,j) where $1\leq i\leq j\leq n$.

Day 2 August 14, 2019

5 In acute-angled triangle ABC, $AB>AC$. Let O, H be the circumcenter and orthocenter of $\triangle ABC$, respectively. The line passing through H and parallel to AB intersects line AC at M, and the line passing through H and parallel to AC intersects line AB at N. L is the reflection of the point H in MN. Line OL and AH intersect at K. Prove that K, M, L, N are concyclic.

6 Let $a_1,a_2,\cdots,a_n(n\geq 2)$ be positive numbers such that $a_1\leq a_2\leq\cdots\leq a_n$. Prove that

$$\sum_{1\leq i<j\leq n}(a_i+a_j)^2\left(\frac{1}{i^2}+\frac{1}{j^2}\right)\geq 4(n-1)\sum_{i=1}^n\frac{a_i^2}{i^2}.$$

7 Prove that for any positive integer k, there exist finitely many sets T satisfying the following two properties: $(1) T$ consists of finitely many prime numbers; (2) $\prod_{p\in T}(p+k)$ is divisible by $\prod_{p\in T} p$.

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8 We call a set S a *good* set if $S=\{x,2x,3x\}(x\neq 0)$. For a given integer $n(n\geq 3)$, determine the largest possible number of the *good* subsets of a set containing n positive integers.
