National Mathematical Olympiad 2005

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by parmenides51, digitalboy1, Kunihiko_Chikaya

– 2nd Round

1 An integer is square-free if it is not divisible by a^2 for any integer $a > 1$. Let S be the set of positive square-free integers. Determine, with justification, the value of

$$\sum_{k \epsilon S} \left[\sqrt{\frac{10^{10}}{k}}\right]$$

where $[x]$ denote the greatest integer less than or equal to x

2 Let G be the centroid of triangle ABC. Through G draw a line parallel to BC and intersecting the sides AB and AC at P and Q respectively. Let BQ intersect GC at E and CP intersect GB at F. If D is midpoint of BC, prove that triangles ABC and DEF are similar

3 Let a, b, c be real numbers satisfying $a < b < c, a + b + c = 6, ab + bc + ac = 9$. Prove that $0 < a < 1 < b < 3 < c < 4$

Let $abc = k$, then a, b, c $(a < b < c)$ are the roots of cubic equation $x^3 - 6x^2 + 9x - k = 0 \Longleftrightarrow x(x-3)^2 = k$

that is to say, a, b, c $(a < b < c)$ are the x-coordinates of the interception of points between $y = x(x-3)^2$ and

$y = k$.

$y = x(x-3)^2$ have local maximuml value of 4 at $x = 1$ and local minimum value of 0 at $x = 3$.

Since the x-coordinate of the interception point between $y = x(x-3)^2$ and $y = 4$ which is the tangent line at

local maximum point $(1, 4)$ is a point $(4, 4)$,Moving the line $y = k$ so that the two graphs $y = x(x-3)^2$ and

$y = k$ have the distinct three interception points,we can find that the range of a, b, c are

$0 < a < 1, 1 < b < 3, 3 < c < 4$,we are done.

4 Place 2005 points on the circumference of a circle. Two points P, Q are said to form a pair of neighbours if the chord PQ subtends an angle of at most 10 degrees at the centre. Find the smallest number of pairs of neighbours.