National Mathematical Olympiad 2006

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– 2nd Round

1 In the triangle $ABC, \angle A = \frac{\pi}{3}, D, M$ are points on the line AC and E, N are points on the line AB such that DN and EM are the perpendicular bisectors of AC and AB respectively. Let L be the midpoint of MN. Prove that $\angle EDL = \angle ELD$

2 Show that any representation of 1 as the sum of distinct reciprocals of numbers drawn from the arithmetic progression $\{2, 5, 8, 11, ...\}$ such as given in the following example must have at least eight terms:

$$1 = \frac{1}{2} + \frac{1}{5} + \frac{1}{8} + \frac{1}{11} + \frac{1}{20} + \frac{1}{41} + \frac{1}{110} + \frac{1}{1640}$$

3 Consider the sequence $p_1, p_2, ...$ of primes such that for each $i \geq 2$, either $p_i = 2p_{i-1} - 1$ or $p_i = 2p_{i-1} + 1$. Show that any such sequence has a finite number of terms.

4 Let n be positive integer. Let $S_1, S_2, \cdots, S_k$ be a collection of $2n$-element subsets of $\{1, 2, 3, 4, ..., 4n-1, 4n\}$ so that $S_i \cap S_j$ contains at most n elements for all $1 \leq i < j \leq k$. Show that

$$k \leq 6^{(n+1)/2}$$

5 Let a, b, n be positive integers. Prove that $n!$ divides

$$b^{n-1}a(a+b)(a+2b)...(a+(n-1)b)$$

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