National Mathematical Olympiad 2007

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–	2nd Round
1	Let $x_1, x_2, \ldots, x_n$ be real numbers satisfying $x_1^2+x_2^2+\ldots+x_n^2=1$. Prove that for every integer $k\geq 2$ there are integers $a_1, a_2, \ldots, a_n$, not all zero, such that $\|a_i\|\leq k-1$ for all i, and $\|a_1x_1+a_2x_2+\ldots+a_nx_n\|\leq\frac{(k-1)\sqrt{n}}{k^n-1}$.
2	Let $n > 1$ be an integer and let $a_1, a_2, ..., a_n$ be n different integers. Show that the polynomial $f(x) = (x-a_1)(x-a_2)\cdot\ldots\cdot(x-a_n)-1$ is not divisible by any polynomial with integer coefficients and of degree greater than zero but less than n and such that the highest power of x has coefficient 1.
3	Let A_1, B_1 be two points on the base AB of an isosceles triangle ABC, with $\angle C > 60^\circ$, such that $\angle A_1CB_1 = \angle ABC$. A circle externally tangent to the circumcircle of $\triangle A_1B_1C$ is tangent to the rays CA and CB at points A_2 and B_2, respectively. Prove that $A_2B_2 = 2AB$.
4	find all functions $f:\mathbb{N}\to\mathbb{N}$ st $f(f(m)+f(n))=m+n \forall m,n\in\mathbb{N}$ related: https://artofproblemsolving.com/community/c6h381298
5	Find the largest integer n such that n is divisible by all positive integers less than $\sqrt[3]{n}$.