National Matematical Olympiad 2019

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–	2nd Round
1	In the acute-angled triangle ABC with circumcircle ω and orthocenter H, points D and E are the feet of the perpendiculars from A onto BC and from B onto AC respecively. Let P be a point on the minor arc BC of ω . Points M and N are the feet of the perpendiculars from P onto lines BC and AC respectively. Let PH and MN intersect at R. Prove that $\angle DMR = \angle MDR$.
2	find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that $f(-f(x) - f(y)) = 1 - x - y \quad \forall x, y \in \mathbb{Z}$
3	A robot is placed at point P on the x-axis but different from $(0,0)$ and $(1,0)$ and can only move along the axis either to the left or to the right. Two players play the following game. Player A gives a distance and B gives a direction and the robot will move the indicated distance along the indicated direction. Player A aims to move the robot to either $(0,0)$ or $(1,0)$. Player B's aim is to stop A from achieving his aim. For which P can A win?
4	Let $p \equiv 2 \pmod 3$ be a prime, k a positive integer and $P(x) = 3x^{\frac{2p-1}{3}} + 3x^{\frac{p+1}{3}} + x + 1$. For any integer n, let $R(n)$ denote the remainder when n is divided by p and let $S = \{0, 1, \cdots, p-1\}$. At each step, you can either (a) replaced every element i of S with $R(P(i))$ or (b) replaced every element i of S with $R(i^k)$. Determine all k such that there exists a finite sequence of steps that reduces S to $\{0\}$. *Proposed by fattypiggy123*
5	In a $m \times n$ chessboard ($m, n \geq 2$), some dominoes are placed (without overlap) with each domino covering exactly two adjacent cells. Show that if no more dominoes can be added to the grid, then at least $2/3$ of the chessboard is covered by dominoes. *Proposed by DVDthe1st, mzy and jjax*