Tournament Of Towns 1982

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by parmenides51

– Spring 1982

– Junior

(015) 1 Find all natural numbers which are divisible by 30 and which have exactly 30 different divisors.

(M Levin)

(016) 2 The lengths of all sides and both diagonals of a quadrilateral are less than 1 metre.
Prove that it may be placed in a circle of radius 0.9 metres.

(017) 3 a) Prove that in an infinite sequence a_k of integers, pairwise distinct and each member greater than 1, one can find 100 members for which $a_k > k$.
b) Prove that in an infinite sequence a_k of integers, pairwise distinct and each member greater than 1 there are infinitely many such numbers a_k such that $a_k > k$.

(A Andjans, Riga)

PS. (a) for juniors (b) for seniors

(018) 4 In a certain country there are more than 101 towns. The capital of this country is connected by direct air routes with 100 towns, and each town, except for the capital, is connected by direct air routes with 10 towns (if A is connected with B, B is connected with A). It is known that from any town it is possible to travel by air to any other town (possibly via other towns). Prove that it is possible to close down half of the air routes connected with the capital, and preserve the capability of travelling from any town to any other town within the country.

(IS Rubanov)

(019) 5 Consider the sequence $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...$
Does there exist an arithmetic progression composed of terms of this sequence
(a) of length 5,
(b) of length greater than 5 (if so, what possible length)?

(G Galperin, Moscow)

– Senior

(020) 1 (a) Prove that for any positive numbers $x_1, x_2, ..., x_k$ ($k > 3$),

$$\frac{x_1}{x_k + x_2} + \frac{x_2}{x_1 + x_3} + ... + \frac{x_k}{x_{k-1} + x_1} \geq 2$$

(b) Prove that for every k this inequality cannot be sharpened, i.e. prove that for every given k it is not possible to change the number 2 in the right hand side to a greater number in such a way that the inequality remains true for every choice of positive numbers $x_1, x_2, ..., x_k$.

(A Prokopiev)

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(021) 2 A square is subdivided into K^2 equal smaller squares. We are given a broken line which passes through the centres of all the smaller squares (such a broken line may intersect itself). Find the minimum number of links in this broken line.

(A Andjans, Riga)

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3 variation of Junior 3

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(022) 4 A polynomial $P(x)$ has unity as the coefficient of its highest power, and has the property that with natural number arguments, it can take all values of form 2^M , where M is a natural number. Prove that the polynomial is of degree 1.

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5 same as Junior 5

– Autumn 1982

– Junior

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(023) 1 There are 36 cards in a deck arranged in the sequence spades, clubs, hearts, diamonds, spades, clubs, hearts, diamonds, etc. Somebody took part of this deck off the top, turned it upside down, and cut this part into the remaining part of the deck (i.e. inserted it between two consecutive cards). Then four cards were taken off the top, then another four, etc. Prove that in any of these sets of four cards, all the cards are of different suits.

(A Merkov, Moscow)

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(024) 2 A number of objects, each coloured in one of two given colours, are arranged in a line (there is at least one object having each of the given colours). It is known that each two objects, between which there are exactly 10 or 15 other objects, are of the same colour. What is the greatest possible number of such pieces?

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(025) 3 Prove that the equation $m!n! = k!$ has infinitely many solutions in which m, n and k are natural numbers greater than unity .

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(026) 4 (a) 10 points dividing a circle into 10 equal arcs are connected in pairs by 5 chords. Is it necessary that two of these chords are of equal length?
(b) 20 points dividing a circle into 20 equal arcs are connected in pairs by 10 chords. Prove that among these 10 chords there are two chords of equal length.

(VV Proizvolov, Moscow)

– Senior

(027) 1 Prove that for all natural numbers greater than 1 $[\sqrt{n}]+[\sqrt[3]{n}]+...+[\sqrt[n]{n}]=[\log_2 n]+[\log_3 n]+ ... +[\log_n n]$.

(VV Kisil)

(028) 2 Does there exist a polyhedron (not necessarily convex) which could have the following complete list of edges? $AB, AC, BC, BD, CD, DE, EF, EG, FG, FH, GH, AH$.

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(029) 3 60 symbols, each of which is either X or O, are written consecutively on a strip of paper. This strip must then be cut into pieces with each piece containing symbols symmetric about their centre, e.g. $O, XX, OXXXXX, XOX$, etc.
(a) Prove that there is a way of cutting the strip so that there are no more than 24 such pieces.
(b) Give an example of such an arrangement of the signs for which the number of pieces cannot be less than 15.
(c) Try to improve the result of (b).

(030) 4 (a) $K_1, K_2, ..., K_n$ are the feet of the perpendiculars from an arbitrary point M inside a given regular n-gon to its sides (or sides produced). Prove that the sum $\overrightarrow{MK_1}+\overrightarrow{MK_2}+...+\overrightarrow{MK_n}$ equals $\frac{n}{2}\overrightarrow{MO}$, where O is the centre of the n-gon.
(b) Prove that the sum of the vectors whose origin is an arbitrary point M inside a given regular tetrahedron and whose endpoints are the feet of the perpendiculars from M to the faces of the tetrahedron equals $\frac{4}{3}\overrightarrow{MO}$, where O is the centre of the tetrahedron.

(VV Prasolov, Moscow)

(031) 5 The plan of a Martian underground is represented by a closed selfintersecting curve, with at most one self-intersection at each point. Prove that a tunnel for such a plan may be constructed in such a way that the train passes consecutively over and under the intersecting parts of the tunnel.