Moscow Mathematical Olympiad 1955

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– tour 1

287 a) The numbers $1, 2, ..., 49$ are arranged in a square table as follows:
https://cdn.artofproblemsolving.com/attachments/5/0/c2e350a6ad0ebb8c728affe0ebb70783baf91
png
Among these numbers we select an arbitrary number and delete from the table the row and the column which contain this number. We do the same with the remaining table of 36 numbers, etc., 7 times. Find the sum of the numbers selected.

b) The numbers $1, 2, ..., k^2$ are arranged in a square table as follows:
https://cdn.artofproblemsolving.com/attachments/2/d/28d60518952c3acddc303e427483211c42cd4
png
Among these numbers we select an arbitrary number and delete from the table the row and the column which contain this number. We do the same with the remaining table of $(k-1)^2$ numbers, etc., k times. Find the sum of the numbers selected.

288 We are given a right triangle ABC and the median BD drawn from the vertex B of the right angle. Let the circle inscribed in $\triangle ABD$ be tangent to side AD at K. Find the angles of $\triangle ABC$ if K divides AD in halves.

289 Consider an equilateral triangle $\triangle ABC$ and points D and E on the sides AB and BCcsuch that $AE = CD$. Find the locus of intersection points of AE with CD as points D and E vary.

290 Is there an integer n such that $n^2 + n + 1$ is divisible by 1955 ?

291 Find all rectangles that can be cut into 13 equal squares.

292 Let a, b, n be positive integers, $b < 10$ and $2^n = 10a + b$.
Prove that if $n > 3$, then 6 divides ab.

293 Consider a quadrilateral $ABCD$ and points K, L, M, N on sides AB, BC, CD and AD, respectively, such that $KB = BL = a, MD = DN = b$ and $KL \nparallel MN$. Find the set of all the intersection points of KL with MN as a and b vary.

294 a) A square table with 49 small squares is filled with numbers 1 to 7 so that in each row and in each column all numbers from 1 to 7 are present. Let the table be symmetric through the main diagonal. Prove that on this diagonal all the numbers $1, 2, 3, ..., 7$ are present.
b) A square table with n^2 small squares is filled with numbers 1 to n so that in each row and

in each column all numbers from 1 to n are present. Let n be odd and the table be symmetric through the main diagonal. Prove that on this diagonal all the numbers $1, 2, 3, ..., n$ are present.

295 Which convex domains on a plane can contain an entire straight line?

296 There are four points A, B, C, D on a circle. Circles are drawn through each pair of neighboring points. Denote the intersection points of neighboring circles by A_1, B_1, C_1, D_1. (Some of these points may coincide with previously given ones.) Prove that points A_1, B_1, C_1, D_1 lie on one circle.

297 Given two distinct nonintersecting circles none of which is inside the other.
Find the locus of the midpoints of all segments whose endpoints lie on the circles.

298 Find all real solutions of the system $\begin{cases} x^3 + y^3 = 1 \\ x^4 + y^4 = 1 \end{cases}$

299 Suppose that primes $a_1, a_2, ..., a_p$ form an increasing arithmetic progression and $a_1 > p$.
Prove that if p is a prime, then the difference of the progression is divisible by p.

300 Inside $\triangle ABC$, there is fixed a point D such that $AC - DA > 1$ and $BC - BD > 1$.
Prove that $EC - ED > 1$ for any point E on segment AB.

301 Given a trihedral angle with vertex O. Find whether there is a planar section ABC such that the angles $\angle OAB$, $\angle OBA$, $\angle OBC$, $\angle OCB$, $\angle OAC$, $\angle OCA$ are acute?

– tour 2

302 Find integer solutions of the equation $x^3 - 2y^3 - 4z^3 = 0$.

303 The quadratic expression $ax^2 + bx + c$ is the 4-th power (of an integer) for any integer x. Prove that $a = b = 0$.

304 The centers O_1, O_2 and O_3 of circles exscribed about $\triangle ABC$ are connected.
Prove that $O_1O_2O_3$ is an acute-angled one.

305 25 chess players are going to participate in a chess tournament. All are on distinct skill levels, and of the two players the one who plays better always wins. What is the least number of games needed to select the two best players?

306 Cut a rectangle into 18 rectangles so that no two adjacent ones form a rectangle.

307 * The quadratic expression $ax^2 + bx + c$ is a square (of an integer) for any integer x. Prove that $ax^2 + bx + c = (dx + e)^2$ for some integers d and e.

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308 * Two circles are tangent to each other externally, and to a third one from the inside. Two common tangents to the first two circles are drawn, one outer and one inner. Prove that the inner tangent divides in halves the arc intercepted by the outer tangent on the third circle.

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309 A point O inside a convex n-gon $A_1A_2...A_n$ is connected with segments to its vertices. The sides of this n-gon are numbered 1 to n (distinct sides have distinct numbers). The segments $OA_1, OA_2, ..., OA_n$ are similarly numbered.
a) For $n = 9$ find a numeration such that the sum of the sides numbers is the same for all triangles $A_1OA_2, A_2OA_3, ..., A_nOA_1$.
b) Prove that for $n = 10$ there is no such numeration.

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310 Let the inequality

$$Aa(Bb + Cc) + Bb(Aa + Cc) + Cc(Aa + Bb) > \frac{ABc^2 + BCa^2 + CAb^2}{2}$$

with given $a > 0, b > 0, c > 0$ hold for all $A > 0, B > 0, C > 0$.
Is it possible to construct a triangle with sides of lengths a, b, c?

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311 Find all numbers a such that

(1) all numbers $[a], [2a], ..., [Na]$ are distinct and

(2) all numbers $\left[\frac{1}{a}\right], \left[\frac{2}{a}\right], ..., \left[\frac{M}{a}\right]$ are distinct.

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312 Given $\triangle\ ABC$, points C_1, A_1, B_1 on sides AB, BC, CA, respectively, such that $\frac{AC_1}{C_1B} = \frac{BA_1}{A_1C} = \frac{CB_1}{B_1A} = \frac{1}{n}$ and points C_2, A_2, B_2 on sides A_1B_1, B_1C_1, C_1A_1 of $\triangle\ A_1B_1C_1$, respectively, such that $\frac{A_1C_2}{C_2B_1} = \frac{B_1A_2}{A_2C_1} = \frac{C_1B_2}{B_2A_1} = n$. Prove that $A_2C_2//AC, C_2B_2//CB, B_2A_2//BA$.

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313 On the numerical line, arrange a system of closed segments of length 1 without common points (endpoints included) so that any infinite arithmetic progression with any difference and any first term has a common point with a segment of the system.

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314 Prove that the equation $x^n - a_1x^{n-1} - a_2x^{n-2} - ... - a_{n-1}x - a_n = 0$, where $a_1 \geq 0, a_2 \geq 0, ..., a_n \geq 0$, cannot have two positive roots.

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315 Five men play several sets of dominoes (two against two) so that each player has each other player once as a partner and two times as an opponent. Find the number of sets and all ways to arrange the players.

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316 Prove that if $\frac{p}{q}$ is an irreducible rational number that serves as a root of the polynomial $f(x) = a_0x^n + a_1x^{n-1} + ... + a_n$ with integer coefficients, then $p - kq$ is a divisor of $f(k)$ for any integer k.

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317 A right circular cone stands on plane P. The radius of the cones base is r, its height is h. A source of light is placed at distance H from the plane, and distance 1 from the axis of the cone. What is the illuminated part of the disc of radius R, that belongs to P and is concentric with the disc forming the base of the cone?

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318 What greatest number of triples of points can be selected from 1955 given points, so that each two triples have one common point?

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319 Consider $\triangle\ A_0B_0C_0$ and points C_1, A_1, B_1 on its sides A_0B_0, B_0C_0, C_0A_0, points C_2, A_2, B_2 on the sides A_1B_1, B_1C_1, C_1A_1 of $\triangle\ A_1B_1C_1$, respectively, etc., so that $\frac{A_0B_1}{B_1C_0} = \frac{B_0C_1}{C_1A_0} = \frac{C_0A_1}{A_1B_0} = k$, $\frac{A_1B_2}{B_2C_1} = \frac{B_1C_2}{C_2A_1} = \frac{C_1A_2}{A_2B_1} = \frac{1}{k^2}$
and, in general, $\frac{A_nB_{n+1}}{B_{n+1}C_n} = \frac{B_nC_{n+1}}{C_{n+1}A_n} = \frac{C_nA_{n+1}}{A_{n+1}B_n} = k^{2n}$ for n even , $\frac{1}{k^{2n}}$ for n odd.
Prove that $\triangle\ ABC$ formed by lines A_0A_1, B_0B_1, C_0C_1 is contained in $\triangle\ A_nB_nC_n$ for any n.
