Tournament Of Towns 1983

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by parmenides51

– Spring 1983

– Junior O Level

(032) O1 A pedestrian walked for 3.5 hours. In every period of one hour's duration he walked 5 kilometres. Is it true that his average speed was 5 kilometres per hour?

(NN Konstantinov, Moscow)

(033) O2 (a) A regular $4k$-gon is cut into parallelograms. Prove that among these there are at least k rectangles.
(b) Find the total area of the rectangles in (a) if the lengths of the sides of the $4k$-gon equal a.

(VV Proizvolov, Moscow)

(034) O3 In Shvambrania there are N towns, every two of which are connected by a road. These roads do not intersect. If necessary, some of them pass over or under others via bridges. An evil magician establishes one-way rules along the roads in such a way that if someone goes out of a certain town he is unable to come back. Prove that
(a) It is possible to establish such rules.
(b) There exists a town from which it is possible to reach any other town, and there exists a town from which it is not possible to go out.
(c) There is one and only one route passing through all towns.
(d) The magician can realise his intention in $N!$ ways.

(LM Koganov, Moscow)

PS. (a),(b),(c) for Juniors, (a),(b),(d) for Seniors

(035) O4 The natural numbers M and K are represented by different permutations of the same digits. Prove that
(a) The sum of the digits of $2M$ equals the sum of the digits of $2K$.
(b) The sum of the digits of $M/2$ equals the sum of the digits of $K/2$ $(M, K$ both even).
(c) The sum of the digits of $5M$ equals the sum of the digits of $5K$.

(AD Lisitskiy)

(036) O5 A version of billiards is played on a right triangular table, with a pocket in each of the three corners, and one of the acute angles being 30^o. A ball is played from just in front of the pocket at the 30^o. vertex toward the midpoint of the opposite side. Prove that if the ball is played hard enough, it will land in the pocket of the 60^o vertex after 8 reflections.

– Junior A Level

A1 same as Junior O1

A2 same as Junior O2

A3 same as Junior A3

(037) A4 (a) An infinite sheet is divided into squares by two sets of parallel lines. Two players play the following game: the first player chooses a square and colours it red, the second player chooses a non-coloured square and colours it blue, the first player chooses a non-coloured square and colours it red, the second player chooses a non-coloured square and colours it blue, and so on. The goal of the first player is to colour four squares whose vertices form a square with sides parallel to the lines of the two parallel sets. The goal of the second player is to prevent him. Can the first player win?

(b) What is the answer to this question if the second player is permitted to colour two squares at once?

(DG Azov)

PS. (a) for Juniors, (a),(b) for Seniors

(038) A5 Prove that in any set of 17 distinct natural numbers one can either find five numbers so that four of them are divisible into the other or five numbers none of which is divisible into any other.

(An established theorem)

– Senior O Level

(039) O1 Numbers from 1 to 1000 are arranged around a circle. Prove that it is possible to form 500 non-intersecting line segments, each joining two such numbers, and so that in each case the difference between the numbers at each end (in absolute value) is not greater than 749.

(AA Razborov, Moscow)

(040) O2 On sides AB, BC and CA of triangle ABC are located points P, M and K, respectively, so that AM, BK and CP intersect in one point and the sum of the vectors $\overrightarrow{AM}, \overrightarrow{BK}$ and $\overrightarrow{CP}$ equals $\overrightarrow{0}$. Prove that K, M and P are midpoints of the sides of triangle ABC on which they are located.

O3 variation of Juniors O3 (034)

(041) O4 There are K boys placed around a circle. Each of them has an even number of sweets. At a command each boy gives half of his sweets to the boy on his right. If, after that, any boy has an odd number of sweets, someone outside the circle gives him one more sweet to make the number even. This procedure can be repeated indefinitely. Prove that there will be a time at which all boys will have the same number of sweets.

(A Andjans, Riga)

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(042) O5 A point is chosen inside a regular k-gon in such a way that its orthogonal projections on to the sides all meet the respective sides at interior points. These points divide the sides into $2k$ segments. Let these segments be enumerated consecutively by the numbers $1, 2, 3, ..., 2k$. Prove that the sum of the lengths of the segments having even numbers equals the sum of the segments having odd numbers.

(A Andjans, Riga)

– Senior A Level

A1 same as Seniors O1 (039)

A2 same as Seniors O2 (040)

A3 same as Seniors O3, variation of Juniors O3 (034)

A4 variation of Juniors A4 (037)

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(043) A5 k vertices of a regular n-gon P are coloured. A colouring is called almost uniform if for every positive integer m the following condition is satisfied:
If M_1 is a set of m consecutive vertices of P and M_2 is another such set then the number of coloured vertices of M_1 differs from the number of coloured vertices of M_2 at most by 1.
Prove that for all positive integers k and n ($k \leq n$) an almost uniform colouring exists and that it is unique within a rotation.

(M Kontsevich, Moscow)

– Autumn 1983

– Junior

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(044) 1 Inside square $ABCD$ consider a point M. Prove that the points of intersection of the medians of triangles ABM, BCM, CDM and DAM form a square.

(V Prasolov)

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(045) 2 Find all natural numbers k which can be represented as the sum of two relatively prime numbers not equal to 1.

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(046) 3 Construct a quadrilateral given its side lengths and the length of the segment joining the midpoints of its diagonals.

(IZ Titovich)

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(047) 4 $a_1, a_2, a_3, ...$ is a monotone increasing sequence of natural numbers. It is known that for any $k, a_{a_k} = 3k$.
a) Find a_{100}.
b) Find a_{1983}.

(A Andjans, Riga)

PS. (a) for Juniors, (b) for Seniors

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(048) 5 N^2 pieces are placed on an $N \times N$ chessboard. Is it possible to rearrange them in such a way that any two pieces which can capture each other (when considered to be knights) after the rearrangement are on adjacent squares (i.e. squares having at least one common boundary point)? Consider two cases:
(a) $N = 3$.
(b) $N = 8$

(S Stefanov)

– Senior

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(049) 1 On sides CB and CD of square $ABCD$ are chosen points M and K so that the perimeter of triangle CMK equals double the side of the square. Find angle MAK.

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(050) 2 Consider all nine-digit numbers, consisting of non-repeating digits from 1 to 9 in an arbitrary order. A pair of such numbers is called "conditional" if their sum is equal to 987654321.
(a) Prove that there exist at least two conditional pairs (noting that (a, b) and (b, a) is considered to be one pair).
(b) Prove that the number of conditional pairs is odd.

(G Galperin, Moscow)

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(051) 3 The centre O of the circumcircle of $\triangle ABC$ lies inside the triangle. Perpendiculars are drawn rom O on the sides. When produced beyond the sides they meet the circumcircle at points K, M and P. Prove that $\overrightarrow{OK} + \overrightarrow{OM} + \overrightarrow{OP} = \overrightarrow{OI}$, where I is the centre of the inscribed circle of $\triangle ABC$.

(V Galperin, Moscow)

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4 variation of Juniors 4 (046)

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(052) 5 A set A of squares is given on a chessboard which is infinite in all directions. On each square of this chessboard which does not belong to A there is a king. On a command all kings may be moved in such a way that each king either remains on its square or is moved to an adjacent square, which may have been occupied by another king before the command. Each square may be occupied by at most one king. Does there exist such a number k and such a way of moving the kings that after k moves the kings will occupy all squares of the chessboard? Consider the following cases:
(a) A is the set of all squares, both of whose coordinates are multiples of 100.
(There is a horizontal line numbered by the integers from $-\infty$ to $+\infty$, and a similar vertical line. Each square of the chessboard may be denoted by two numbers, its coordinates with respect to these axes.)
(b) A is the set of all squares which are covered by 100 fixed arbitrary queens
(i.e. each square covered by at least one queen).

Remark:
If A consists of just one square, then $k = 1$ and the required way is the following:
all kings to the left of the square of A make one move to the right.
