Tournament Of Towns 1984

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by parmenides51

– Spring 1984

– Junior O Level

(053) O1 The price of 175 Humpties is more than the price of 125 Dumpties but less than that of 126 Dumpties.
Prove that you cannot buy three Humpties and one Dumpty for
(a) 80 cents.
(b) 1 dollar.

(S Fomin, Leningrad)

PS. (a) for Juniors , (a),(b) for Seniors

(054) O2 In the convex pentagon $ABCDE, AE = AD, AB = AC$, and angle CAD equals the sum of angles AEB and ABE. Prove that segment CD is double the length of median AM of triangle ABE.

(055) O3 Consider the $4(N - 1)$ squares on the boundary of an N by N array of squares. We wish to insert in these squares $4(N - 1)$ consecutive integers (not necessarily positive) so that the sum of the numbers at the four vertices of any rectangle with sides parallel to the diagonals of the array (in the case of a degenerate rectangle, i.e. a diagonal, we refer to the sum of the two numbers in its corner squares) are one and the same number.
Is this possible? Consider the cases
(a) $N = 3$
(b) $N = 4$
(c) $N = 5$

(VG Boltyanskiy, Moscow)

(056) O4 The product of the digits of the natural number N is denoted by $P(N)$ whereas the sum of these digits is denoted by $S(N)$. How many solutions does the equation $P(P(N))+P(S(N))+S(P(N)) + S(S(N)) = 1984$ have?

(057) O5 An infinite squared sheet is given, with squares of side length 1. The distance between two squares is defined as the length of the shortest path from one of these squares to the other if moving between them like a chess rook (measured along the trajectory of the centre of the rook). Determine the minimum number of colours with which it is possible to colour the sheet (each square being given a single colour) in such a way that each pair of squares with distance between them equal to 6 units is given different colours. Give an example of such a colouring and prove that using a smaller number of colours we cannot achieve this goal.

(AG Pechkovskiy, IV Itenberg)

– Junior A Level

A1 same as Junior O5 (057)

(058) A2 In a ballroom dance class 15 boys and 15 girls are lined up in parallel rows so that 15 couples are formed. It so happens that the difference in height between the boy and the girl in each couple is not more than 10 cm. Prove that if the boys and the girls were placed in each line in order of decreasing height, then the difference in height in each of the newly formed couples would still be at most 10 cm.

(AG Pechkovskiy, Moscow)

A3 same as Junior O3 (055)

(059) A4 Show how to cut an isosceles right triangle into a number of triangles similar to it in such a way that every two of these triangles is of different size.

(AV Savkin)

(060) A5 The two pairs of consecutive natural numbers $(8, 9)$ and $(288, 289)$ have the following property: in each pair, each number contains each of its prime factors to a power not less than 2. Prove that there are infinitely many such pairs.

(A Andjans, Riga)

– Senior O Level

O1 same as Junior O2 (054)

(061) O2 Six altitudes are constructed from the three vertices of the base of a tetrahedron to the opposite sides of the three lateral faces. Prove that all three straight lines joining two base points of the altitudes in each lateral face are parallel to a certain plane.

(IF Sharygin, Moscow)

(062) O3 From a squared sheet of paper of size $29 \times 29, 99$ pieces, each a 2×2 square, are cut off (all cutting is along the lines bounding the squares). Prove that at least one more piece of size 2×2 may be cut from the remaining part of the sheet.

(S Fomin, Leningrad)

(063) O4 Prove that, for any natural number n, the graph of any increasing function $f : [0, 1] \to [0, 1]$ can be covered by n rectangles each of area whose sides are parallel to the coordinate axes. Assume that a rectangle includes both its interior and boundary points.
(a) Assume that $f(x)$ is continuous on $[0, 1]$.
(b) Do not assume that $f(x)$ is continuous on $[0, 1]$.

(A Andjans, Riga)

PS. (a) for O Level, (b) for A Level

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(064) O5 (a) On each square of a squared sheet of paper of size 20×20 there is a soldier. Vanya chooses a number d and Petya moves the soldiers to new squares in such a way that each soldier is moved through a distance of at least d (the distance being measured between the centres of the initial and the new squares) and each square is occupied by exactly one soldier. For which d is this possible?
(Give the maximum possible d, prove that it is possible to move the soldiers through distances not less than d and prove that there is no greater d for which this procedure may be carried out.)
(b) Answer the same question as (a), but with a sheet of size 21×21.

(SS Krotov, Moscow)

– Senior A Level

A1 same as Seniors O5 (064)

A2 same as Seniors O2 (061)

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(065) A3 An infinite (in both directions) sequence of rooms is situated on one side of an infinite hallway. The rooms are numbered by consecutive integers and each contains a grand piano. A finite number of pianists live in these rooms. (There may be more than one of them in some of the rooms.) Every day some two pianists living in adjacent rooms (the Arth and ($k + 1$)st) decide that they interfere with each others practice, and they move to the ($k-1$)st and ($k+2$)nd rooms, respectively. Prove that these moves will cease after a finite number of days.

(VG Ilichev)

A4 variation of Seniors O4 (063)

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(066) A5 Let $p(n)$ be the number of partitions of the natural number n into natural summands. The diversity of a partition is by definition the number of different summands in it. Denote by $q(n)$ the sum of the diversities of all the $p(n)$ partitions of n.
(For example, $p(4) = 5$, the five distinct partitions of 4 being $4, 3+1, 2+2, 2+1+1, 1+1+1+1$, and $g(4) = 1 + 2 + 1 + 2 + 1 = 7$.)
Prove that, for all natural numbers n,

(a) $q(n) = 1 + P(1) + P(2) + p(3) + ... + p(n-1)$,
(b) $q(n) < \sqrt{2n}p(n)$.

(AV Zelevinskiy, Moscow)

– Autumn 1984

– Junior

– Training

(067) T1 In triangle ABC the bisector of the angle at B meets AC at D and the bisector of the angle at C meets AB at E. These bisectors intersect at O and the lengths of OD and OE are equal. Prove that either $\angle BAC = 60^o$ or triangle ABC is isosceles.

(068) T2 A village is constructed in the form of a square, consisting of 9 blocks , each of side length ℓ, in a 3×3 formation . Each block is bounded by a bitumen road . If we commence at a corner of the village, what is the smallest distance we must travel along bitumen roads , if we are to pass along each section of bitumen road at least once and finish at the same corner?

(Muscovite folklore)

(069) T3 Find all solutions of $2^n + 7 = x^2$ in which n and x are both integers . Prove that there are no other solutions.

(070) T4 Inside a rectangle is inscribed a quadrilateral, which has a vertex on each side of the rectangle. Prove that the perimeter of the inscribed quadrilateral is not smaller than double the length of a diagonal of the rectangle.

(V. V . Proizvolov , Moscow)

(071) T5 Prove that among 18 consecutive three digit numbers there must be at least one which is divisible by the sum of its digits.

– Main

1 same as Training 1 (O67)

2 same as Training 2 (O68)

(072) 3 On a plane there is a finite set of M points, no three of which are collinear . Some points are joined to others by line segments, with each point connected to no more than one line segment . If we have a pair of intersecting line segments AB and CD we decide to replace them with AC and BD, which are opposite sides of quadrilateral $ABCD$. In the resulting system of segments we decide to perform a similar substitution, if possible, and so on . Is it possible that such substitutions can be carried out indefinitely?

(V.E. Kolosov)

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(073) 4 Six musicians gathered at a chamber music festival . At each scheduled concert some of these musicians played while the others listened as members of the audience . What is the least number of such concerts which would need to be scheduled in order to enable each musician to listen , as a member of the audience, to all the other musicians?

(Canadian origin)

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(074) 5 On the Island of Camelot live 13 grey, 15 brown and 17 crimson chameleons . If two chameleons of different colours meet , they both simultaneously change colour to the third colour (e .g . if a grey and brown chameleon meet each other they both change to crimson) . Is it possible that they will eventually all be the same colour?

(V . G . Ilichev)

– Senior

– Training

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(075) T1 In convex hexagon $ABCDEF, AB$ is parallel to CF, CD is parallel to BE and EF is parallel to AD. Prove that the areas of triangles ACE and BDF are equal .

T2 same as Junior Training 2 (068)

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(076) T3 In $\triangle ABC, \angle ABC = \angle ACB = 40^o$. BD bisects $\angle ABC$, with D located on AC. Prove that $BD + DA = BC$.

T4 same as Junior Training 4 (071)

T5 same as Junior 5 (074)

– Main

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1 same as Training 1 (075)

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(077) 2 A set of numbers $a_1, a_2, ..., a_{100}$ is obtained by rearranging the numbers $1, 2, ..., 100$. Form the numbers $b_1 = a_1$ $b_2 = a_1 + a_2$ $b_3 = a_1 + a_2 + a_3$
... $b_{100} = a_1 + a_2 + ... + a_{100}$
Prove that among the remainders on dividing the numbers by $100, 11$ of them are different .

(L . D . Kurlyandchik , Leningrad)

(078) 3 We are given a regular decagon with all diagonals drawn. The number "$+1$ " is attached to each vertex and to each point where diagonals intersect (we consider only internal points of intersection). We can decide at any time to simultaneously change the sign of all such numbers along a given side or a given diagonal . Is it possible after a certain number of such operations to have changed all the signs to negative?

4 same as Junior 4 (073)

(079) 5 A 7×7 square is made up of 16 1×3 tiles and 1 1×1 tile. Prove that the 1×1 tile lies either at the centre of the square or adjoins one of its boundaries .